End-of-term Exam

EE1C11 “Linear Circuits A”

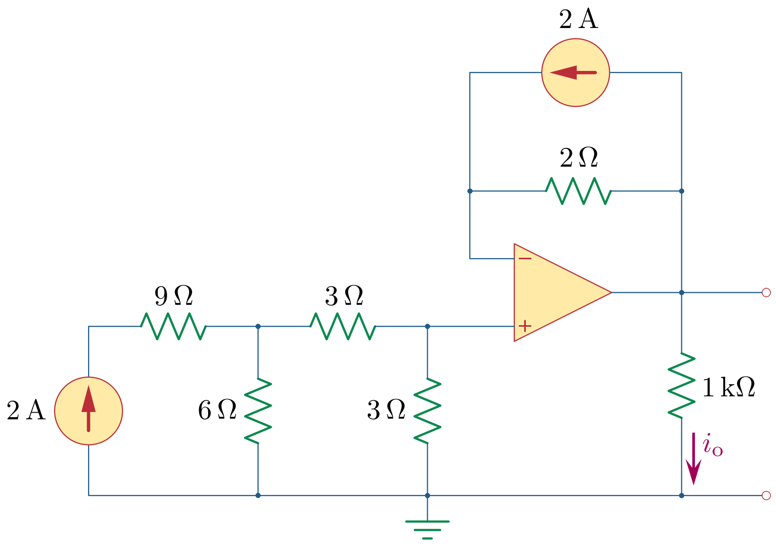
- This exam consists of 4 exercises.
- Each exercise accounts for **10 points**; the total number of points to be obtained is **40**. The exam grade is obtained by dividing the total number of points by 4, rescaling linearly the result to the 1-10 scale and rounding off to 1 decimal.
- **Each exercise must be solved on a separate double-sheet.** Writing more solutions on the same sheet may result in only one of the solutions being graded!
- Indicate your name and study number on **each** submitted sheet. **You must hand in (blank) signed sheets even for the exercises that you do not handle.**
- Students benefitting of the “Extra Time” (ET) rule are entitled to a 20 minutes extension of their exam provided they produce the relevant supporting document.
- Should any question not be completely clear, you are allowed to ask the instructors in the exam hall; the answer will be confined to rephrasing the text of the exercise such that to make it more intelligible.
- Should a part of an exercise depend on a previous result, mistakes made at a previous step will only be penalised once.
- Give your solution as completely as possible and never state numerical results without indicating how you derived them. **Simply stating numerical results will yield no points.**
- **When requested, fill in the measure units for all calculated quantities.** This holds for intermediate results but definitely for the final ones.
- Write clearly and avoid messy solutions. Should errors occur in your solution, cross the erroneous part out and give clear indications on where the correct solution resumes.
- For this exam you are allowed to use:
    1. a simple calculator – programmable and graphic calculators are explicitly prohibited;
    2. a handwritten, double-sided A4 sheet with formulas.
- The text of this exam is offered only in English. Inasmuch as possible, instructors will assist you with the Dutch translation of formulations that you may have difficulties to understand.

The Linear Circuits team wishes you a lot of success!

- Take a new double-sheet -

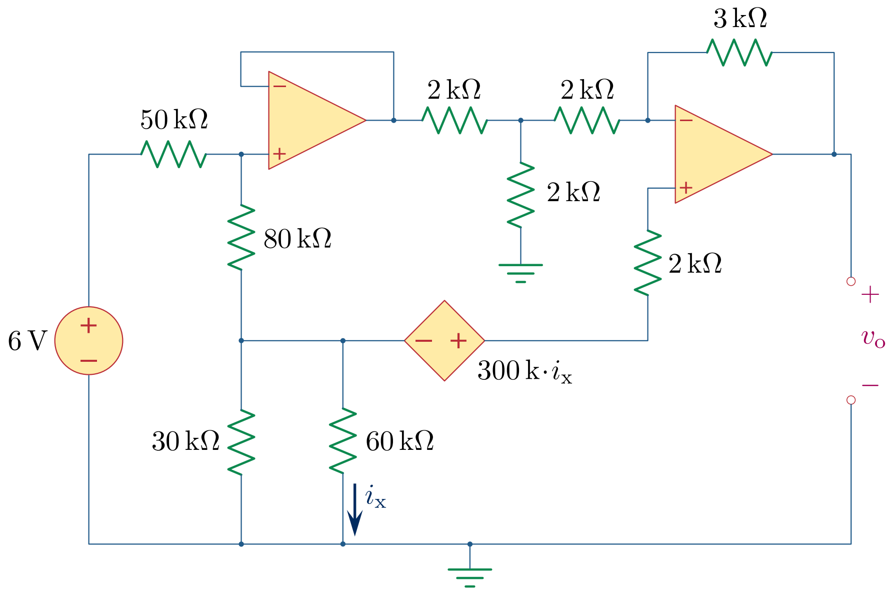
Exercise 1

Consider the circuits in the figure below:



a) Calculate the current i_o in the circuit, by assuming the op amp to be ideal. (6 points)

Now consider the circuit in the figure below:



b) Calculate the output voltage v_o, assuming both op amps to be ideal. (4 points)

Indicate the measure units for all calculated quantities. Show all steps in your reasoning and never give numerical results without justification.

Solution

Sub-point (a)

$$V_+ = 3 \cdot i_2 = 3 \cdot \frac{6}{6+(3+3)} \cdot 2 = 3\,V$$

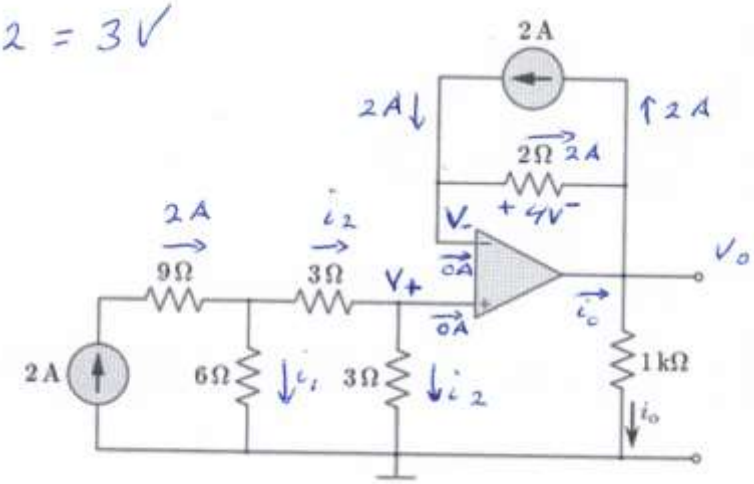
$$V_- = V_+ = 3\,V$$

$$V_- - V_o = 4\,V$$

$$3 - V_o = 4\,V$$

$$\Rightarrow V_o = 3 - 4 = -1\,V$$

$$i_o = \frac{-1}{1k\Omega} = -1\,mA$$

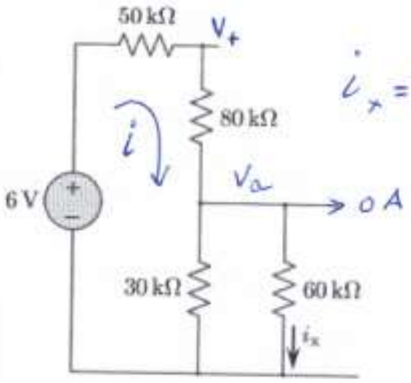


Sub-point (b)

$$i = \frac{6}{50k + 80k + \underbrace{30k /\!/ 60k}_{20k}} = \frac{6}{150k} = 40\,\mu A.$$

$V_+ = i \cdot (80k + 20k) = 4\,V$ or as Voltage divider.

$$V_+ = 6 \cdot \frac{80k+20k}{80k+20k+50k} = 4\,V$$



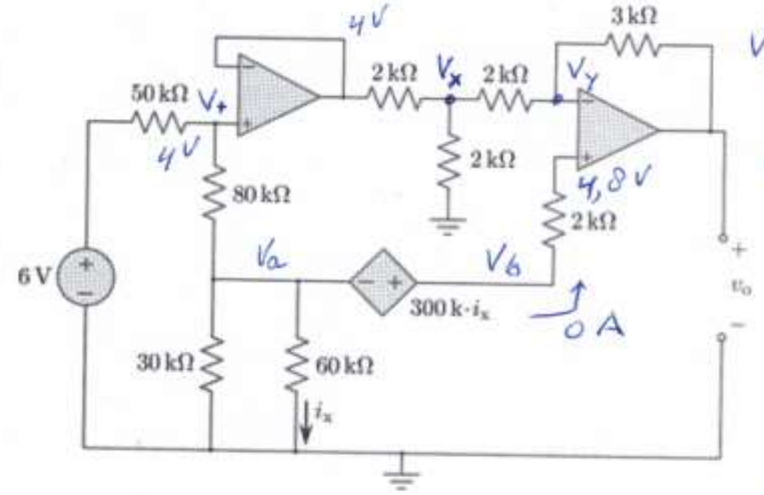
$$i_x = i \cdot \frac{30k}{30k+60k} = \frac{i}{3} = \frac{40}{3}\,\mu A,$$

$$V_a = 40\,\mu A \cdot 20k = 0{,}8\,V$$

$$V_b = V_a + 300 \cdot 10^3 \cdot \frac{40}{3} \cdot 10^{-6} = 0{,}8 + 4 = 4{,}8\,V$$

KCL @ V_x: $\frac{V_x - 4}{2k} + \frac{V_x}{2k} + \frac{V_x - 4{,}8}{2k} = 0$

$$V_x = \frac{8{,}8}{3}$$



KCL @ V_y:

$$\frac{V_y - V_x}{2k} + \frac{V_y - V_o}{3k} = 0$$

$$V_y = 4{,}8\,V$$

$$V_x = \frac{8{,}8}{3}\,V$$

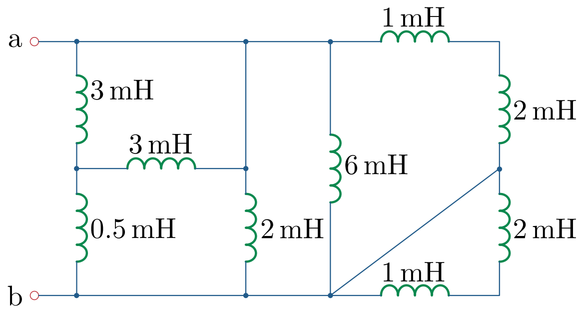
$\Rightarrow$

$$V_o = 7{,}6\,V$$

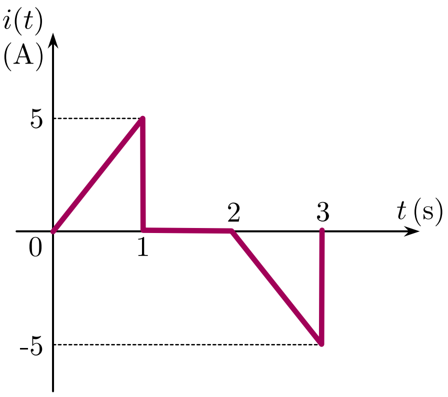
- Take a new double-sheet -

Exercise 2

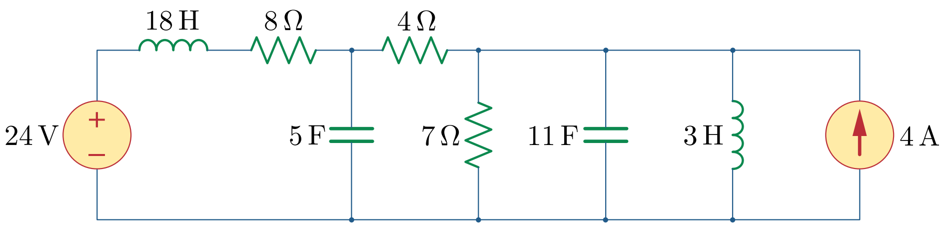
a) Consider the circuit in the figure below. Calculate the equivalent inductance seen from terminals a–b. (3 points)



b) The following ***current*** waveform is applied to a 10 mF capacitance. By taking $v_C(0) = 0\,\text{V}$, calculate the capacitor voltage $v_C(t)$ for $0 < t < 3\,(\text{s})$. (3 points)



c) Now consider the circuit in the figure below. Calculate the total stored energy in ***all*** capacitances and inductances, under DC conditions. (4 points)

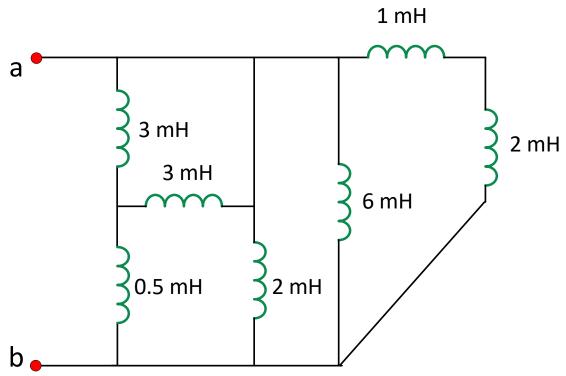


Indicate the measure units for all calculated quantities. Show all steps in your reasoning and never give numerical results without justification.

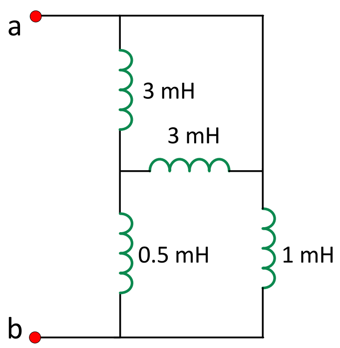
Solution

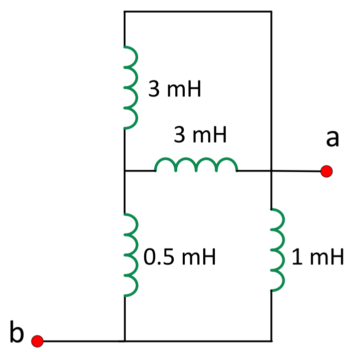
Sub-point (a)

After removing the short circuited part:



$$(1\ mH + 2\ mH)\|6\ mH\|2\ mH = 1mH$$





$$3\ mH\|3\ mH = 1.5\ mH$$

$$(1.5\ mH + 0.5\ mH)\Big\|1\ mH = \frac{2}{3}\ mH$$

Sub-point (b)

We have:

$$v(t) = V(0) + \frac{1}{C}\int_0^t idt$$

$$for\ 0 \le t \le 1: \qquad v(t) = 100\int_0^t 5t\ dt = 250t^2\ V$$

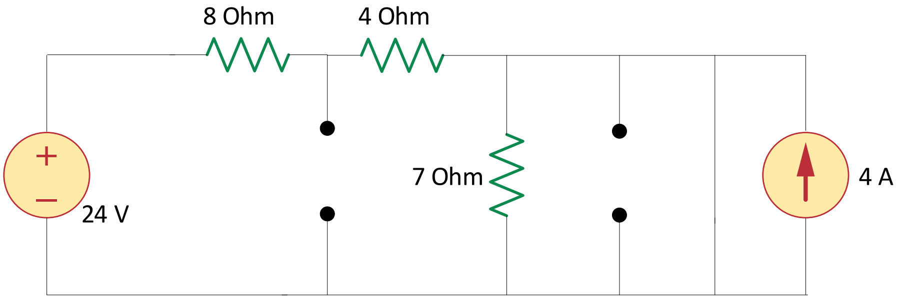
$$for\ 1 \le t \le 2: \qquad v(t) = 250 + 100\int_1^t 0\ dt = 250\ V$$

$$for\ 2 \le t \le 3: \qquad v(t) = 250 + 100\int_2^t -5(t-2)dt = 250 + 100\int_2^t -5t + 10\ dt$$

$$= 250 + 100 \times \left(-\frac{5}{2}t^2 + 10t\right)\Bigg|_2^t = 250 - 250t^2 + 1000t - 1000\ V$$

Sub-point (c)

Redrawing the circuit under DC (inductors shorted and capacitors open circuited):



All 4 A of the current source will flow through the shorted 3 H inductor. No current flows through 7 Ohm resistor. The voltage of 24 V source is over 8+4 Ohm resistor which generates 2 A. therefore in total 6 A is passing through 3 H inductor. 2 A is passing through 18 H inductor. Voltage across 11 F capacitor is zero, and voltage across 5 F capacitor is $\frac{4}{4+8}24 = 8\,V$

Total energy stored in all capacitances and inductances is:

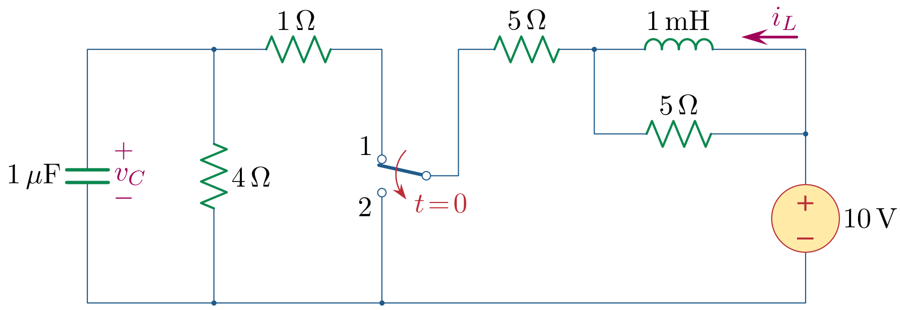
$$w = \frac{1}{2}Li^2 + \frac{1}{2}Li^2 + \frac{1}{2}CV^2 + \frac{1}{2}CV^2$$

$$w = \frac{1}{2}3 \times 36 + \frac{1}{2}18 \times 4 + \frac{1}{2}7 \times 0 + \frac{1}{2}5 \times 64 = 54 + 36 + 0 + 160 = 250\,J$$

- Take a new double-sheet -

Exercise 3

Consider the circuit in the figure below, in which the switch has been in position 1 for a long time and is instantaneously moved to position 2 at $t = 0$ (s):



a) Calculate the current $i_L(0^+)$. (2 points)

b) Calculate the voltage $v_C(0^+)$. (1 point)

c) Calculate the current $i_L(t)$ for $t > 0$. (4 points)

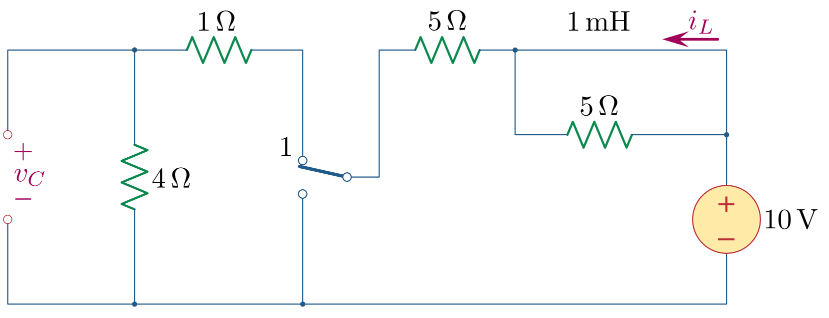
d) Calculate the voltage $v_C(t)$ for $t > 0$. (3 points)

Indicate the measure units for all calculated quantities. Show all steps in your reasoning and never give numerical results without justification.

Solution

Sub-point (a)

The circuit corresponding to $t < 0$ is given below:



From it, is clear that the current $i_L(t)$ follows directly from Ohm's law as

$$i_L = \frac{10}{5+1+4} = 1\,(\mathrm{A})$$

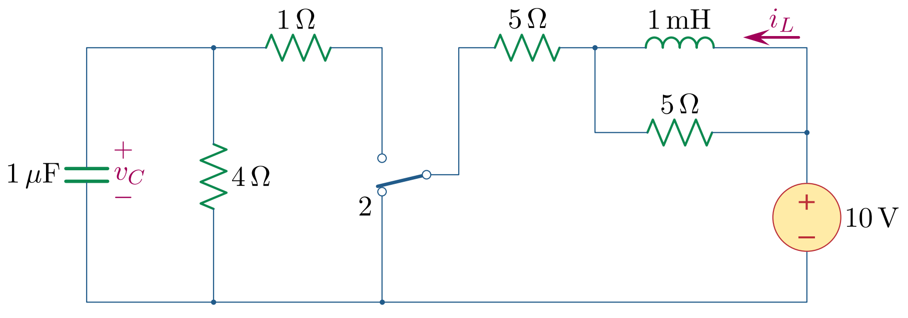
From the continuity condition is then found that $i_L(0^-) = i_L(0^+) = 1(\mathrm{A})$.

Sub-point (b)

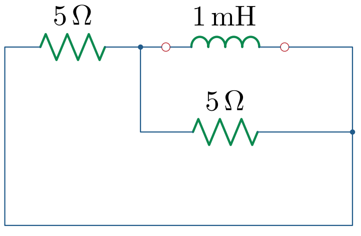
By using the result from the previous point, it readily follows that $v_C(0^-) = 4i_L(0^-) = 4(\mathrm{V})$ which, via the continuity condition, yields $v_C(0^-) = v_C(0^+) = 4(\mathrm{V})$.

Sub-point (c)

The circuit corresponding to $t > 0$ is given below:



It is clear that the new circuit consists of two independent first order circuits. We now focus on the circuit's right section, consisting of one 1 mH inductance, 2 × 5Ω resistances and the 10 V independent voltage source. For the final value we observe that in steady-state DC the inductance again becomes a short circuit implying that $i_L(\infty) = \frac{10}{5} = 2\,(\mathrm{A})$. For calculating the time constant we passivize the circuit, the resulting schematic being given in the figure at the right.



The Thévenin resistance at the inductance's terminals consists of the two 5Ω resistances in parallel, the equivalent resistance being then $R = R_{\text{Th}} = 2.5\Omega$. With this value, the circuit's time constant is $\tau = \dfrac{L}{R} = \dfrac{1\text{m}}{2.5} = 4{\cdot}10^{-4}\,(\text{s})$. By now making use of the initial/final values formula it is found that

$$i_L(t) = i_L(\infty) + \left[i_L(0^+) - i_L(\infty)\right]\exp(-t/\tau) = 2 - \exp(-2500t)\,(\text{A})$$

which is our final result.

Sub-point (d)

We now focus on the circuit's left section, consisting of the 1 μF capacitance and the 4Ω resistance. Evidently, the loose-end 1Ω resistance plays no role. Since this circuit has no sources, the final value will obviously be zero, namely $v_C(\infty) = 0\,\text{V}$. Since this circuit only contains a capacitance and a resistance, the circuit's time constant will be $\tau = RC = 1{\cdot}10^{-6} \times 4 = 4{\cdot}10^{-6}\,(\text{s})$ and by making use of the initial/final values formula it is found that

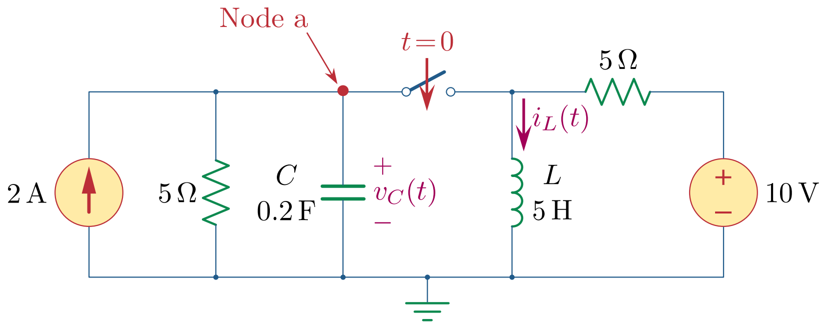
$$v_C(t) = v_C(\infty) + \left[v_C(0^+) - v_C(\infty)\right]\exp(-t/\tau) = 4\exp(-250000t)\,(\text{V})$$

which is our final result.

- Take a new double-sheet -

Exercise 4

Consider the following circuit, in which the switch was open for a long time and is closed at $t = 0\,(\mathrm{s})$:



a) Determine $v_C(0^+)$ and $i_L(0^+)$. (1 point)

b) Determine $v_C(\infty)$ and $i_L(\infty)$. (1 point)

c) Determine $\dfrac{\mathrm{d}v_C(0^+)}{\mathrm{d}t}$ and $\dfrac{\mathrm{d}i_L(0^+)}{\mathrm{d}t}$. (2 points)

d) Apply KCL at node a to derive an expression for the integro-differential equation (not a second-order differential equation) for $v_C(t)$. (2 points)

e) Derive the characteristic equation for this circuit. (1 point)

f) Is this circuit Overdamped, Critically damped or Underdamped? Motivate your answer. (1 point)

g) Determine $i_L(t)$ for $t \geqslant 0$. (2 points)

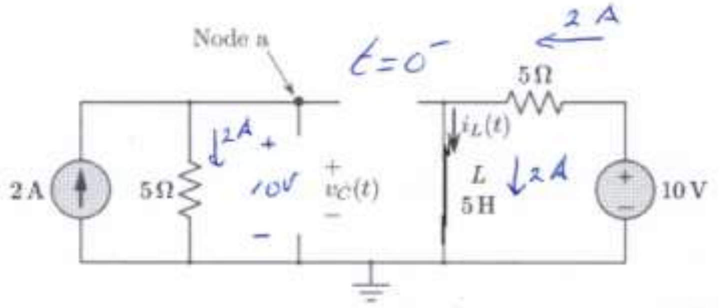
Indicate the measure units for all calculated quantities. Show all steps in your reasoning and never give numerical results without justification.

Solution

Sub-point (a)

$V_C(0^-) = V_C(0^+) = 2A \cdot 5\Omega = 10V$

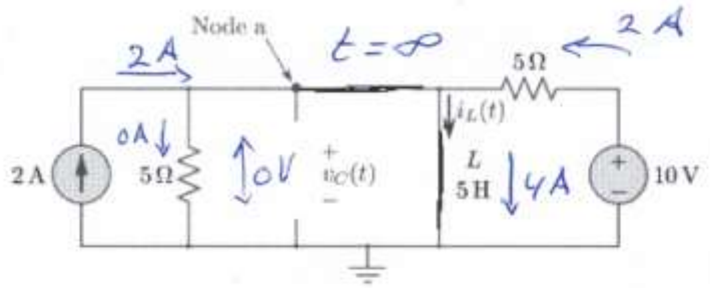
$i_L(0^-) = i_L(0^+) = \frac{10V}{5\Omega} = 2A.$



Sub-point (b)

$V_C(\infty) = V_L(\infty) = 0V$

$i_L(\infty) = 2 + \frac{10}{5} = 4A.$



Sub-point (c)

at $t=0^+$ we have.

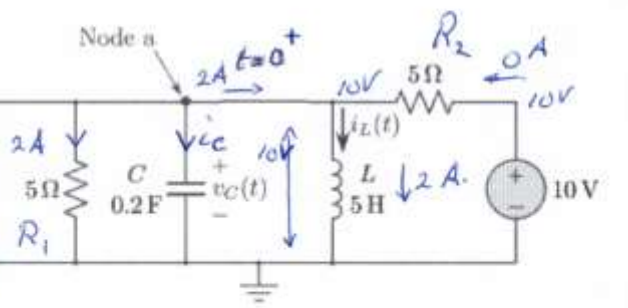
$V_{R_1} = 10V\ (V_C(0^+)) \rightarrow i_{R_1} = 2A$

$V_{R_2} = 10 - 10 = 0V \rightarrow i_{R_2} = 0A.$

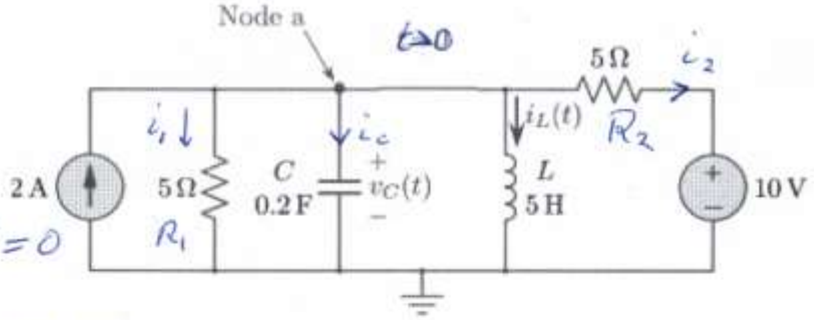
$i_L(0^+) = 2A \Rightarrow i_L(0^+) = -i_C(0^+)$

$i_C = C\frac{dV}{dt} \xrightarrow{at\ t=0} \frac{dV_C(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{-2}{C} = -10V/s$

$\frac{di_L(0^+)}{dt} = \frac{V_L(0^+)}{L} = \frac{V_C(0^+)}{L} = \frac{10}{5} = 2A/s$



Sub-point (d)



KVl @ a :

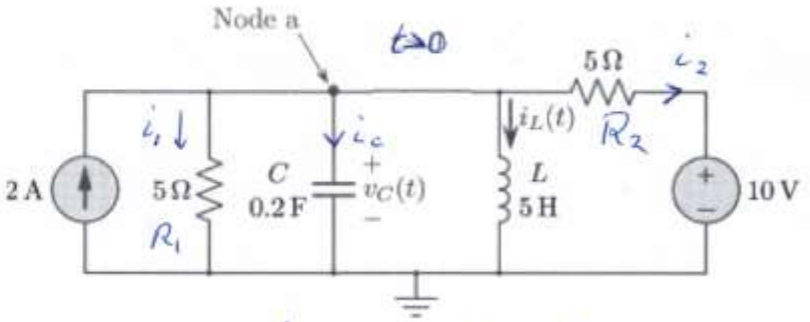
$$-2+i_1+i_c+i_L+i_2=0$$

$$-2+\frac{V}{R_1}+C\frac{dV}{dt}+\frac{1}{L}\int_{-\infty}^{t} V_L\,dt+\frac{V-10}{R_2}=0$$

$$R_1=R_2=R$$

$$\Rightarrow 2\frac{V}{R}+C\frac{dV}{dt}+\frac{1}{L}\int_{-\infty}^{t} V_L\,dt=2+\frac{10}{5}=4$$

Sub-point (e)



$$\frac{d^2V}{dt^2}+\frac{2}{CR}\frac{dV}{dt}+\frac{V}{LC}=0 \Rightarrow$$

$$s^2+\frac{2}{RC}s+\frac{1}{LC}=0 \Leftrightarrow s^2+2s+1=0 \quad (s^2+2\alpha s+\omega_0^2=0) \Rightarrow \alpha=1;\ \omega_0=1$$

Sub-point (f)

$-\alpha\pm\sqrt{\alpha^2-\omega_0^2}=-\alpha\pm\sqrt{0}=-\alpha\pm 0$ repeated solution in -1
critical damped $\Rightarrow$ natural response : $(A_1+A_2t)e^{-\alpha t}$

Sub-point (g)

$$i_L(t)=i_{ss}(t)+i_N(t)=4+(A_1+A_2t)e^{-t}$$

$$i_L(0)=2=4+A_1 \Rightarrow A_1=-2 \qquad \frac{di_L(0^+)}{dt}=2=-A_1+A_2 \Rightarrow A_2=0.$$

$$i_L(t)=4-2e^{-t}\ (A) \quad t\geq 0.$$