End-of-term exam

EE1C1 "Linear Circuits A"

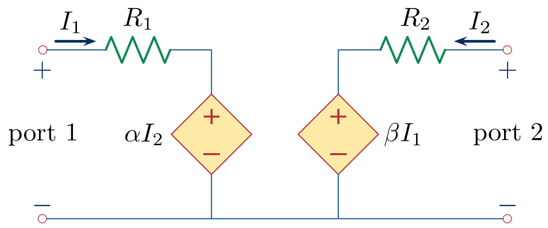
- This exam consists of 4 exercises.
- Each exercise accounts for **10 points**; the total number of points to be obtained is **40**. The exam grade is obtained by dividing the total number of points by 4, rescaling linearly the result to the 1-10 scale and rounding off to 1 decimal.
- **Each exercise must be solved on a separate double-sheet.** Writing more solutions on the same sheet may result in only one of the solutions being graded!
- Indicate your name and study number on **each** submitted sheet. **You must hand in (blank) signed sheets even for the exercises that you do not handle.**
- Students benefitting of the "Extra Time" (ET) rule are entitled to a 20 minutes extension of their exam provided they produce the relevant supporting document.
- Should any question not be completely clear, you are allowed to ask the instructors in the exam hall; the answer will be confined to rephrasing the text of the exercise such that to make it more intelligible.
- Should a part of an exercise depend on a previous result, mistakes made at a previous step will only be penalised once.
- Give your solution as completely as possible and never state numerical results without indicating how you derived them. **Simply stating numerical results will yield no points.**
- **When requested, fill in the measure units for all calculated quantities.** This holds for intermediate results but definitely for the final ones.
- Write clearly and avoid messy solutions. Should errors occur in your solution, cross the erroneous part out and give clear indications on where the correct solution resumes.
- For this exam you are allowed to use:
    i. a simple calculator – programmable and graphic calculators are explicitly prohibited;
    ii. a handwritten, double-sided A4 sheet with formulas.
- The text of this exam is offered only in English. Inasmuch as possible, instructors will assist you with the Dutch translation of formulations that you may have difficulties to understand.

The Linear Circuits team wishes you a lot of success!

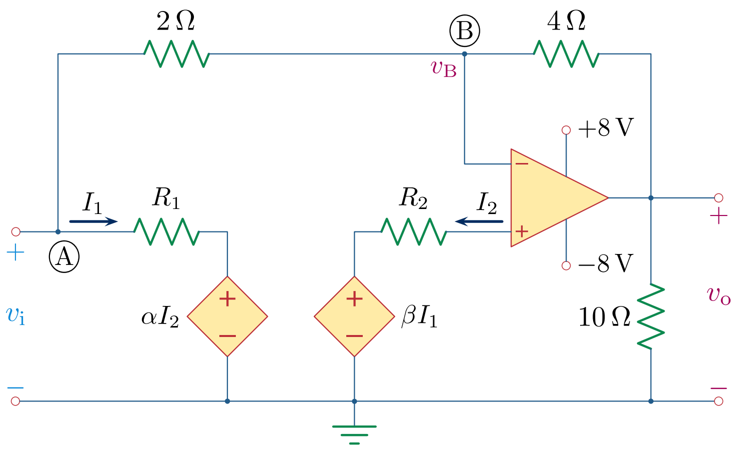
- Take a new double-sheet -

Exercise 1

Any linear and resistive two-ports component can be modelled using the circuit below:



Consider that such a component is connected to an ideal op-amp as shown below:



a) Calculate the ratio of voltage at node B over the input voltage (v_B/v_i). State your answer in terms of $R1$, $R2$, α and β. (3 points)

Hint: In this subpoint, and in the following ones, some of the parameters $R1$, $R2$, α and β may be missing in the requested expressions.

b) Calculate the gain v_o/v_i of this op-amp circuit. State your answer in terms of $R1$, $R2$, α and β. (3 points)

c) What conditions are required to achieve a gain smaller than -1? State your answer in terms of $R1$, $R2$, α and β. (2 points)

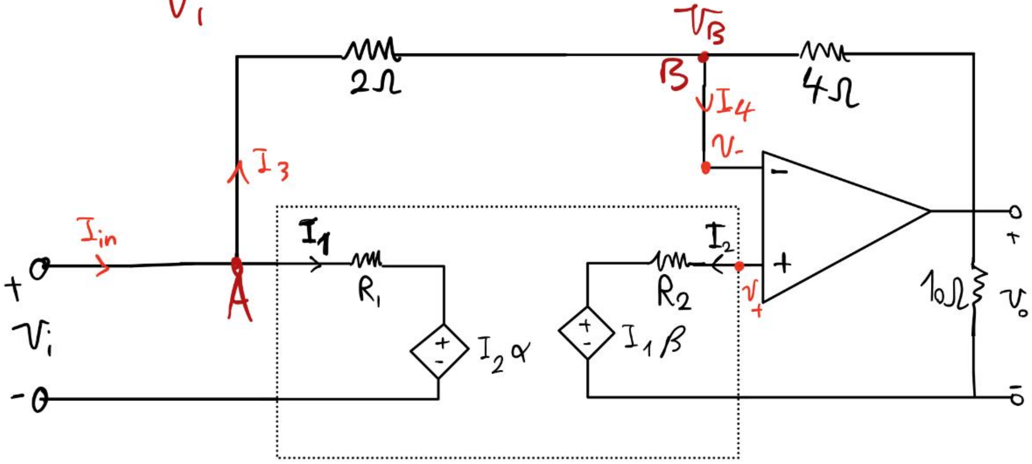
d) Assume that the input voltage is $v_i = 2\,\text{V}$. Considering the two voltage sources supplying power to this op-amp (±8 V), what conditions are required to prevent this op-amp from saturating (op-amp stays in the linear region of operation) while still ensuring a gain smaller than -1? State your answer in terms of $R1$, $R2$, α and β (2 points)

Indicate the measure units for all calculated quantities. Show all steps in your reasoning and never give numerical results without justification.

Solution

Sub-point (a)

$$\frac{v_B}{V_i} = ?$$



looking at our ideal op amp:

$V_- = V_+$

$I_4 = 0$ and $I_2 = 0$ $\xrightarrow{\text{therefore}}$  $I_2\alpha$ is short cir.

We have these equations to consider:

[1] $v_{in} = R_1 I_1 \longrightarrow I_1 = \frac{v_{in}}{R_1}$

[2] $V_+ = I_1 \beta$

$\rightarrow V_+ = v_{in} \frac{\beta}{R_1}$ [3]

$$\boxed{\frac{V_+}{v_{in}} = \frac{\beta}{R_1}}$$

Sub-point (b)

gain? $\frac{v_o}{v_{in}} = ?$

[4] $v_{in} = 2 I_3 + V_-$

[5] $V_{in} = 6 I_3 + V_o \longrightarrow I_3 = \frac{V_{in} - V_o}{6}$ [6]

[6] in [4] using [3] $\longrightarrow v_{in} = \frac{v_{in}}{3} - \frac{v_o}{3} + \frac{v_{in} \beta}{R_1}$

$$\Rightarrow v_o = v_{in}\left(\frac{3\beta}{R_1} + 1 - 3\right)$$

$$\Rightarrow \frac{v_o}{v_{in}} = \frac{3\beta}{R_1} - 2$$

Sub-point (c)

Inverting amplification:

$$\frac{v_o}{v_{in}} < -1 \Rightarrow \frac{3\beta}{R_1} - 2 < -1$$

$$\Rightarrow \frac{\beta}{R_1} < \frac{1}{3}$$

Sub-point (d)

not saturating while invert amp:

I) $\frac{\beta}{R_1} < \frac{1}{3}$

II) $-8 \leq v_{in}\left(\frac{3\beta}{R_1} - 2\right) \leq +8$

$-4 \leq \frac{3\beta}{R_1} - 2 \leq 4 \rightarrow -2 \leq \frac{3\beta}{R_1} \leq 6$

$-\frac{2}{3} \leq \frac{\beta}{R_1} \leq 2$ and $\frac{\beta}{R_1} < \frac{1}{3}$ then $\boxed{-\frac{2}{3} \leq \frac{\beta}{R_1} < \frac{1}{3}}$

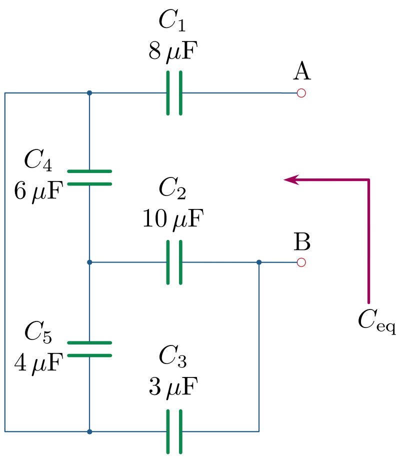
- Take a new double-sheet -

Exercise 2

If the current through a $20\,\mathrm{mH}$ inductor is given by $i(t) = 2e^{-2t} + 2t^2\ \mathrm{A}$, then:

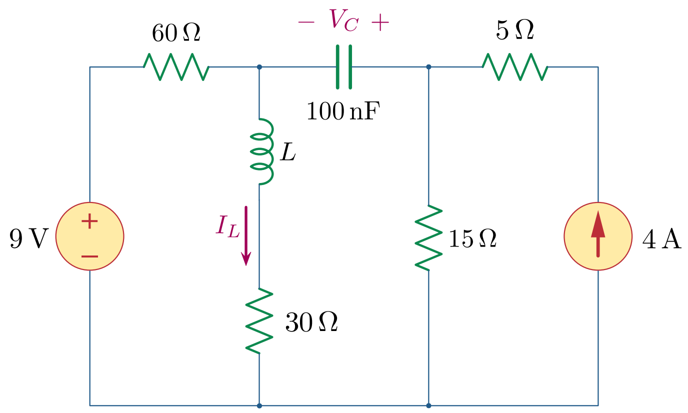
a) Calculate the voltage across the inductor at time $t = 1\mathrm{s}$. (2 points)

Now consider the circuit in the figure below:



b) Calculate the equivalent capacitance C_{eq} at the terminal A–B. (2 points)

Finally, consider now the following circuit under steady state conditions:



c) Calculate the current I_L through the inductor and the voltage V_C across the capacitor. (3 points)

d) Calculate the value of the inductance L so that the energy stored in the capacitor is equal to ***three times*** the energy stored in the inductor. (3 points)

Indicate the measure units for all calculated quantities. Show all steps in your reasoning and never give numerical results without justification.

Solution

Sub-point (a)

$$i(t) = 2e^{-2t} + 2t^2 \, A \qquad L = 20 \, mH$$

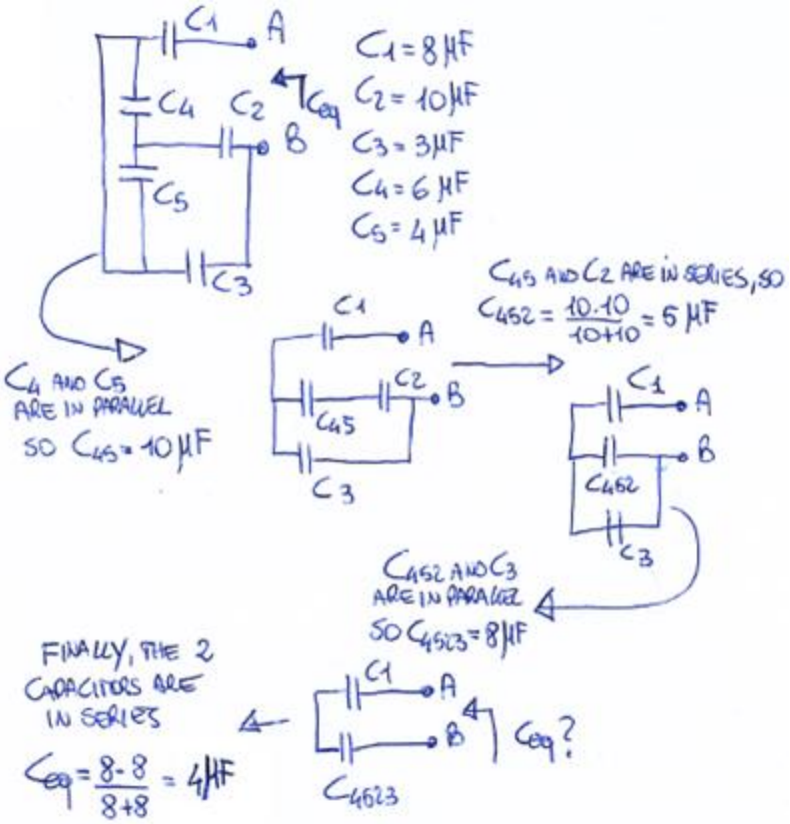
$$V(t) = L\frac{di(t)}{dt} = L\left[2(-2)e^{-2t} + 4t\right] = 80 \cdot 10^{-3}\left[-e^{-2t} + 1\right] V$$

$$V(1) = 80 \cdot 10^{-3}\left[1 - e^{-2}\right]$$

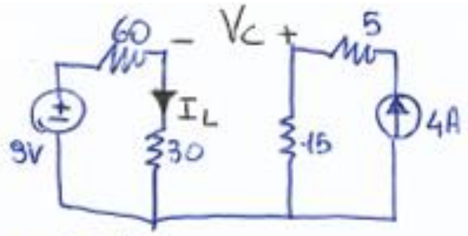
$$80 \cdot 10^{-3}\left[1 - e^{-2}\right] = 69.1732 \cdot 10^{-3} V$$

$$= 0.069 V = 69 mV$$

Sub-point (b)



Sub-point (c)



$$I_L = \frac{V}{\text{TOTAL } R} = 9/(60+30) = \underline{0.1A}$$

$$V_C = V_C^+ - V_C^- \Rightarrow V_C^+ = 4 \cdot 15 = 60V \text{ (OHM'S LAW)}$$

$$V_C^- = \frac{9 \cdot 30}{60+30} = \frac{9 \cdot 30}{90} = 3V \text{ (VOLTAGE DIVIDER)}$$

$$V_C = 57V$$

Sub-point (d)

$$\frac{1}{2} C V_C^2 = 3 \cdot \frac{1}{2} L I_L^2 \Rightarrow L = \frac{C V_C^2}{3 I_L^2} =$$

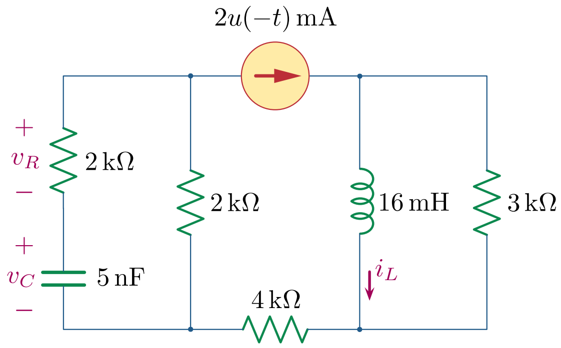
$$= \frac{100 \cdot 10^{-9} \cdot 57 \cdot 57}{3 \cdot 0.1 \cdot 0.1} =$$

$$= 0.0108 H \simeq 11 mH$$

- Take a new double-sheet -

Exercise 3

Consider the circuit in the figure below:



a) Calculate $v_C(0^+)$ and $i_L(0^+)$. (2 points)

b) Redraw the circuit that applies for $t > 0$. (1 point)

c) Calculate $v_R(0^+)$. (2 points)

d) Calculate $v_R(t)$ and $i_L(t)$ for $t \longrightarrow \infty$. (1 point)

e) Determine the expression of $i_L(t)$ for $t > 0$. (2 points)

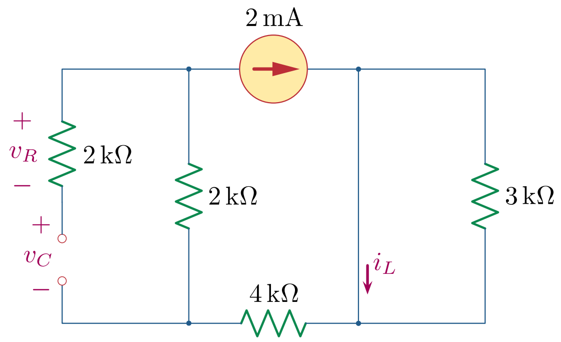
f) Determine the expression of $v_R(t)$ for $t > 0$. (2 points)

Indicate the measure units for all calculated quantities. Show all steps in your reasoning and never give numerical results without justification.

Solution

Sub-point (a)

The circuit for $t < 0$ is:



The short-circuit at the right captures the complete current of the current source, implying that $i_L(0^-) = 2\,\text{mA}$. Furthermore, that current can only go through the 2kΩ resistance not in series with the capacitance. Since the voltage drop on the 2kΩ resistance in series with the capacitance is zero, it follows that

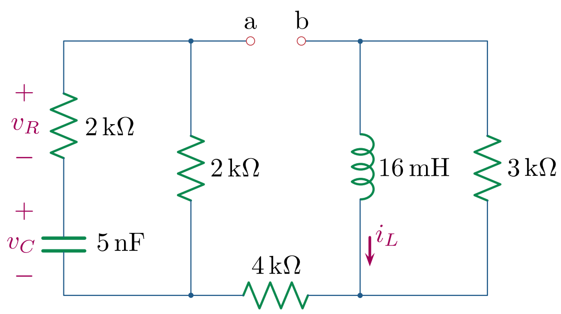
$$v_C(0^-) = -2\text{k} \cdot 2\text{m} = -4\,\text{V}$$

Mind the relation between the direction of $i_L(0^-)$ and the polarity of $v_C(0^-)$! Based on the continuity conditions it is then concluded that:

$$v_C(0^+) = v_C(0^-) = -4\,\text{V}$$
$$i_L(0^+) = i_L(0^-) = 2\,\text{mA}$$

Sub-point (b)

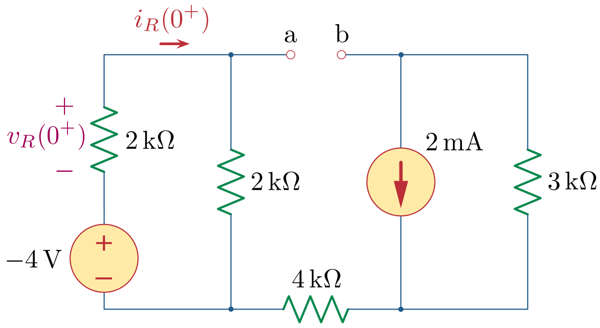
The circuit for $t > 0$ is:



Note that the zero current source translates into an ***open circuit***, effectively separating the initial circuit into ***two independent sections***, each of which has ***only one*** reactive element!

Sub-point (c)

For calculating $v_R(0^+)$, we redraw the circuit at $t = 0^+$:



The right loop with the inductance was only drawn for consistency, it has no effect on $v_R(0^+)$. The direction of $i_R(0^+)$ in the figure is based on the polarity of the voltage source replacing the charged capacitance. It is now clear that

$$v_R(0^+) = -i_R(0^+) \cdot 2\text{k} = -\frac{v_C(0^+)}{2\text{k} + 2\text{k}} \cdot 2\text{k} = -\frac{-4}{4\text{k}} \cdot 2\text{k} = 2\,\text{V}$$

Sub-point (d)

For calculating $v_R(t)$ and $i_L(t)$ for $t \to \infty$ it is observed that both loops containing reactive elements have no sources for $t > 0$. It is then obvious that

$$v_R(\infty) = 0\,\text{V}$$
$$i_L(\infty) = 0\,\text{mA}$$

Sub-point (e)

From the schematic of the circuit for $t > 0$ it can be inferred that the Thévenin resistance at the terminals of the 16mH inductance is the 3kΩ resistance. The correspondent time constant is then

$$\tau_L = \frac{L}{R_{\text{Th},L}} = \frac{16\text{m}}{3\text{k}} = 16/3\,\mu\text{s} = 5.333\,\mu\text{s}$$

and, via the initial-final values formula, it is concluded that

$$i_L(t) = i_L(\infty) + \Big[i_L(0^+) - i_L(\infty)\Big] e^{-t/\tau_L} = 2\,e^{-3/16 \cdot 10^6 t} = 2\,e^{-1.875 \cdot 10^5 t}\,\text{mA}$$

Sub-point (f)

From the schematic of the circuit for $t > 0$ it can be inferred that the Thévenin resistance at the terminals of the 5nF capacitance consists of the two 2kΩ resistances in series. The correspondent time constant is then

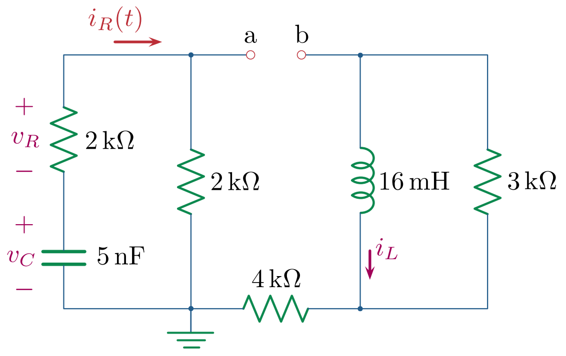
$$\tau_R = R_{\text{Th},C}\, C = 4\text{k} \cdot 5\text{n} = 20\,\mu\text{s}$$

and, via the initial-final values formula, it is concluded that

$$v_R(t) = v_R(\infty) + \Big[v_R(0^+) - v_R(\infty)\Big] e^{-t/\tau_R} = 2\,e^{-1/20 \cdot 10^6 t} = 2\,e^{-5 \cdot 10^4 t}\,\text{V}$$

Note

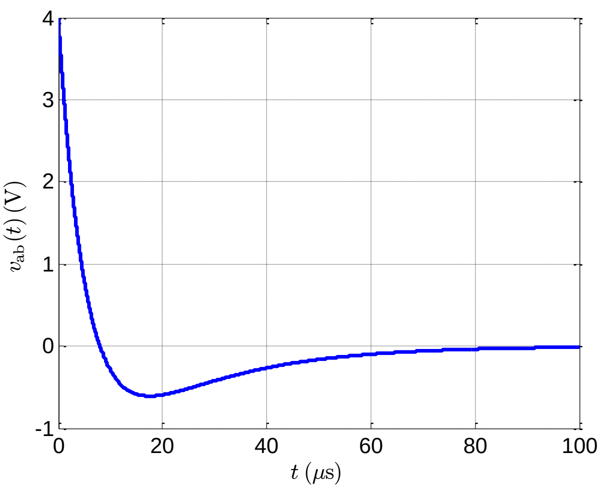
Although the circuit splits up into two independent, typical 1st order circuits, the fact that the circuit, as a whole, contains two reactive elements should manifest itself somewhere in some 2nd order behaviour. And, indeed, the voltage $v_{\mathrm{ab}}(t)$ between the terminals left open by the turned-off current source does present this behaviour. To demonstrate that, the circuit for $t > 0$ is redrawn as:



It is an easy exercise to check that the current through the 4kΩ resistance at the bottom is zero (it belongs to no closed loop in the circuit or, alternatively, one can apply KCL) and, thus, the voltage drop across it is also zero. It then follows that the two loops are referenced to the same ground (see figure) and, thus

$$\begin{aligned} v_{\mathrm{ab}}(t) &= v_{\mathrm{a}} - v_{\mathrm{b}} = i_R(t)\,2\mathrm{k} - \Big[-i_L(t)\,3\mathrm{k}\Big] = -\frac{v_R(t)}{2\mathrm{k}}\,2\mathrm{k} + i_L(t)\,3\mathrm{k} \\ &= 6\,e^{-1.875\cdot 10^5 t} - 2\,e^{-5\cdot 10^4 t}\ (\mathrm{V}) \end{aligned}$$

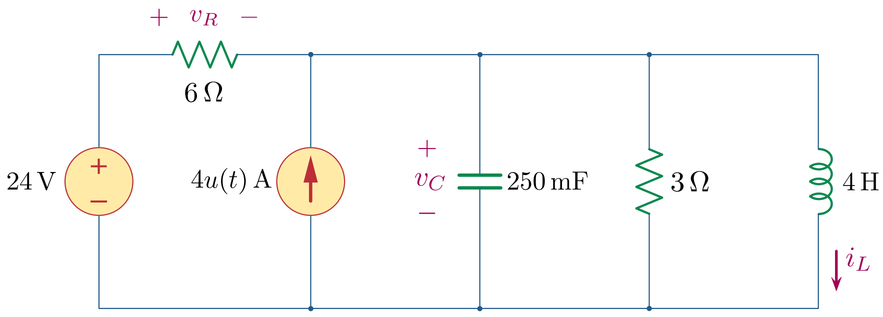
which is a typical, overdamped response (with zero final steady-state value). The relevant time-domain signature is:



- Take a new double-sheet -

Exercise 4

Consider the circuit in the figure below:



a) Calculate $i_L(0^+)$, $v_C(0^+)$, $v_R(0^+)$, and $i_L(\infty)$, $v_C(\infty)$, $v_R(\infty)$. (3 points)

b) Calculate $\frac{\mathrm{d}v_C(0^+)}{\mathrm{d}t}$, $\frac{\mathrm{d}i_L(0^+)}{\mathrm{d}t}$, $\frac{\mathrm{d}v_R(0^+)}{\mathrm{d}t}$. (3 points)

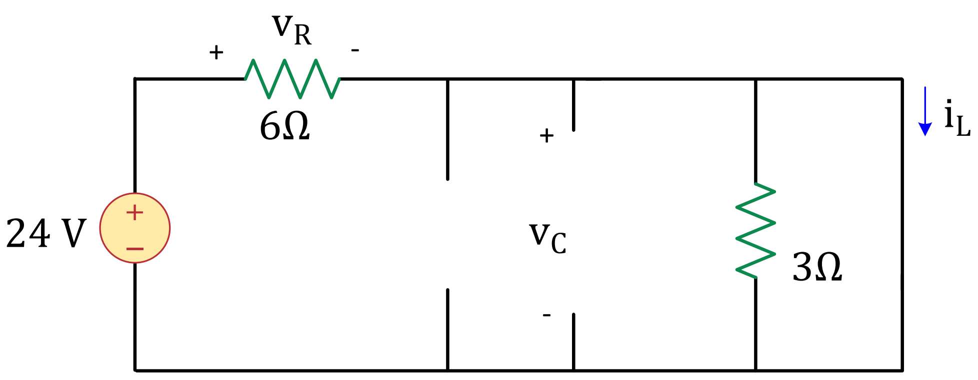
c) Calculate $i_L(t)$ for $t > 0$. (4 points)

Indicate the measure units for all calculated quantities. Show all steps in your reasoning and never give numerical results without justification.

Solution

Sub-point (a)

The circuit at $t = 0^-$ looks like below

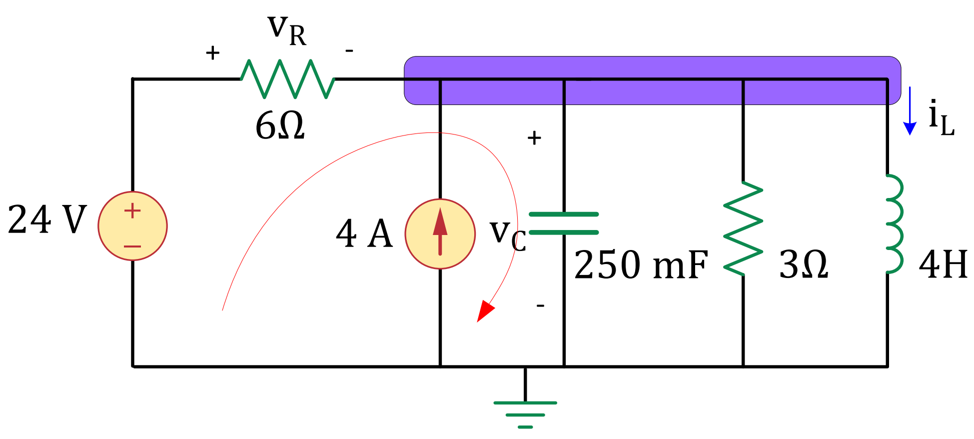


Simple inspection shows that:

$$i_L(0^-) = \frac{24}{6} = 4\ A \rightarrow i_L(0^+) = 4\ A$$

$$v_C(0^-) = 0\ V \rightarrow v_C(0^+) = 0\ V$$

The circuit at $t = 0^+$ looks like below

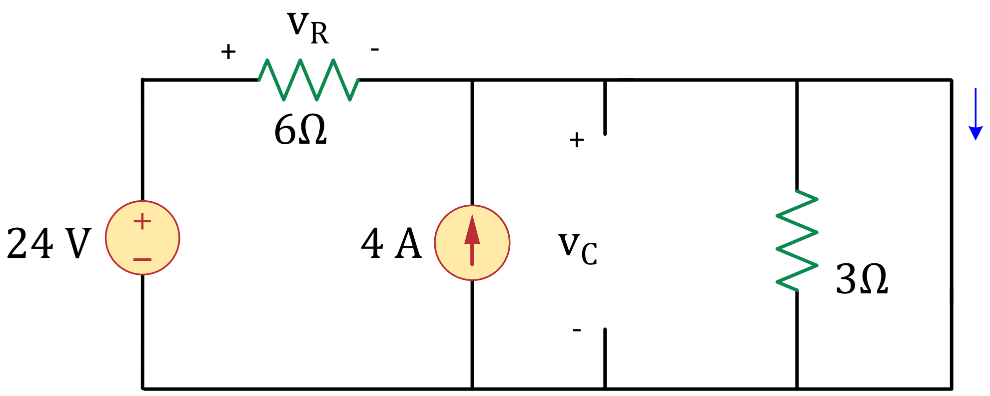


At time $t = 0^+$ we can write kvl to find the voltage of the resistor:

$$-24 + v_R(0^+) + v_C(0^+) = 0$$

$$v_R(0^+) = 24\ V$$

The circuit at $t = \infty$ looks like below



$$i_L(\infty) = \frac{24}{6} + 4 = 8\,A$$

$$v_C(\infty) = 0\,V$$

$$v_R(\infty) = 24\,V$$

Sub-point (b)

We account for the circuit at $t = 0^+$ given above. Writing kcl for the purple node at $t = 0^+$ we have:

$$i_C(0^+) + \frac{v_C(0^+)}{3} + i_L(0^+) - 4 - \frac{v_R(0^+)}{6} = 0 \to i_C(0^+) = 4 \to C\frac{dv_C(0^+)}{dt} = 4 \to \frac{dv_C(0^+)}{dt} = 16\,V/s$$

$$v_C(0^+) = v_L(0^+) = L\frac{di_L(0^+)}{dt} \to \frac{di_L(0^+)}{dt} = 0\,A/s$$

Writing kvl for the shown loop:

$$-24 + v_R(t) + v_C(t) = 0$$

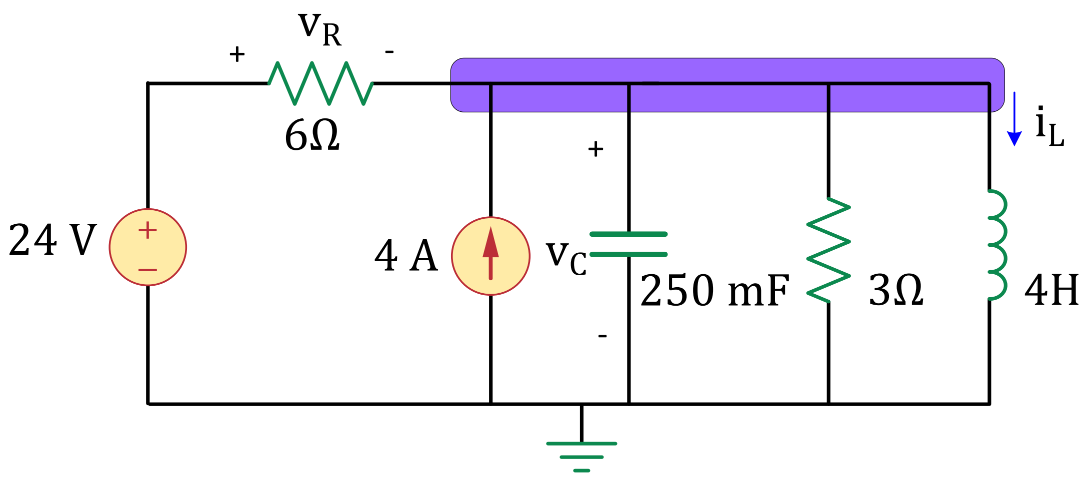
Taking derivative of both sides:

$$\frac{dv_R(t)}{dt} = -\frac{dv_C(t)}{dt}$$

$$\frac{dv_R(0^+)}{dt} = -\frac{dv_C(0^+)}{dt} = -16\,V/s$$

Sub-point (c)

Writing kcl for the purple node at $t > 0$ we have:



$$i_L + \frac{v_L}{3} + i_C - 4 - \frac{24 - v_L}{6} = 0$$

$$i_L + \frac{4}{3}\frac{di_L}{dt} + C\frac{dv_C}{dt} + \frac{4}{6}\frac{di_L}{dt} = 8$$

$$i_L + \frac{4}{3}\frac{di_L}{dt} + CL\frac{d^2 i_L}{dt^2} + \frac{4}{6}\frac{di_L}{dt} = 8$$

$$\frac{d^2 i_L}{dt^2} + 2\frac{di_L}{dt} + i_L = 8$$

$$s^2 + 2s + 1 = 0 \rightarrow s = -1, s = -1$$

$$i_L(t) = 8 + (A_1 + A_2 t)e^{-t}$$

$$i_L(0^+) = 8 + A_1 \rightarrow A_1 = -4$$

$$\frac{di_L}{dt} = -e^{-t}(A_1 + A_2 t) + A_2 e^{-t}$$

$$\frac{di_L(0^+)}{dt} = 4 + A_2 \rightarrow A_2 = -4$$

$i_L(t) = 8 - (4 + 4t)e^{-t}$ for t>0