End-of-term Exam

EE1C21 “Linear Circuits B”

Place:
Date:
Time:

- This exam consists of 4 exercises.
- Each exercise accounts for **10 points**; the total number of points to be obtained is **40**. The exam grade is obtained by dividing the total number of points by 4, rescaling linearly the result to the 1-10 scale and rounding off to 1 decimal.
- **Each exercise must be solved on a separate double-sheet.** Writing more solutions on the same sheet may result in only one of the solutions being graded!
- Indicate your name and study number on **each** submitted sheet. **You must hand in (blank) signed sheets even for the exercises that you do not handle.**
- Students benefitting of the “Extra Time” (ET) rule are entitled to a 20 minutes extension of their exam provided they produce the relevant supporting document.
- Should any question not be completely clear, you are allowed to ask the instructors in the exam hall; the answer will be confined to rephrasing the text of the exercise such that to make it more intelligible.
- Should a part of an exercise depend on a previous result, mistakes made at a previous step will only be penalised once.
- Give your solution as completely as possible and never state numerical results without indicating how you derived them. **Simply stating numerical results will yield no points.**
- **When requested, fill in the measure units for all calculated quantities.** This holds for intermediate results but definitely for the final ones.
- Write clearly and avoid messy solutions. Should errors occur in your solution, cross the erroneous part out and give clear indications on where the correct solution resumes.
- For this exam you are allowed to use:
  i. a simple calculator – programmable and graphing calculators are explicitly prohibited;
  ii. a handwritten, double-sided A4 sheet with formulas.
- The text of this exam is offered only in English. Inasmuch as possible, instructors will assist you with the Dutch translation of formulations that you may have difficulties to understand.

The Linear Circuits team wishes you a lot of success!

Laplace transform pairs.*

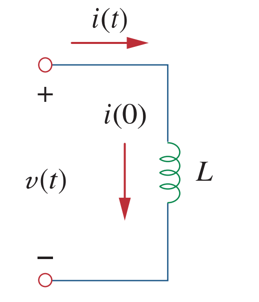
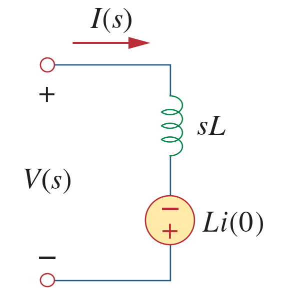
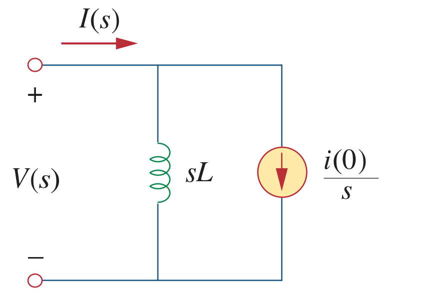
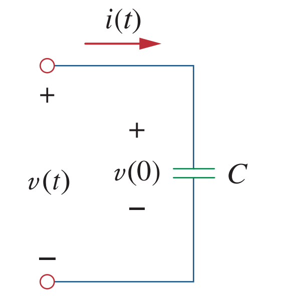
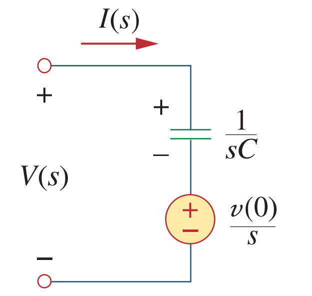
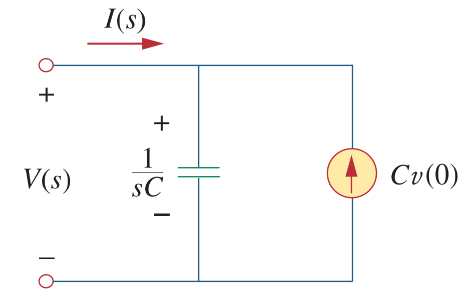
$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t+\theta)$	$\frac{s\sin\theta+\omega\cos\theta}{s^2+\omega^2}$
$\cos(\omega t+\theta)$	$\frac{s\cos\theta-\omega\sin\theta}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

*Defined for $t \geq 0$; $f(t) = 0$, for $t < 0$.

Properties of the Laplace transform.

Property	$f(t)$	$F(s)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Scaling	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
Time shift	$f(t-a)u(t-a)$	$e^{-as}F(s)$
Frequency shift	$e^{-at}f(t)$	$F(s+a)$
Time differentiation	$\frac{df}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2f}{dt^2}$	$s^2F(s) - sf(0^-) - f'(0^-)$
	$\frac{d^3f}{dt^3}$	$s^3F(s) - s^2f(0^-) - sf'(0^-) - f''(0^-)$
	$\frac{d^nf}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \cdots - f^{(n-1)}(0^-)$
Time integration	$\int_0^t f(x)\,dx$	$\frac{1}{s}F(s)$
Frequency differentiation	$tf(t)$	$-\frac{d}{ds}F(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(s)\,ds$
Time periodicity	$f(t) = f(t+nT)$	$\frac{F_1(s)}{1-e^{-sT}}$
Initial value	$f(0)$	$\lim_{s\to\infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s\to 0} sF(s)$
Convolution	$f_1(t) * f_2(t)$	$F_1(s)F_2(s)$

Laplace-domain equivalent circuits for inductances and capacitances

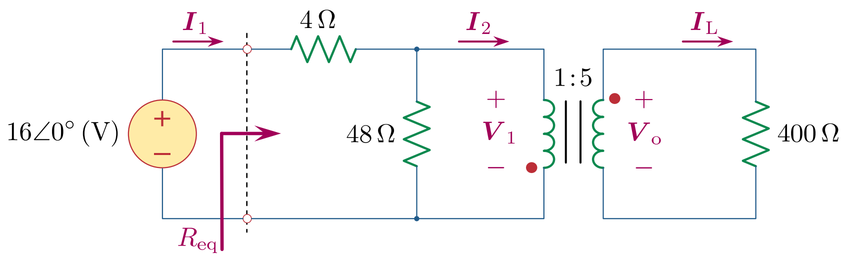
Time-domain circuit	Thévenin-type equivalent	Norton-type equivalent
		
		

Initial-conditions voltage/current values: $v(0) = v(0^-)$ and $i(0) = i(0^-)$.

- Take a new double-sheet -

Exercise 1

Consider the circuit in the figure below, in which the transformer ***is ideal***:

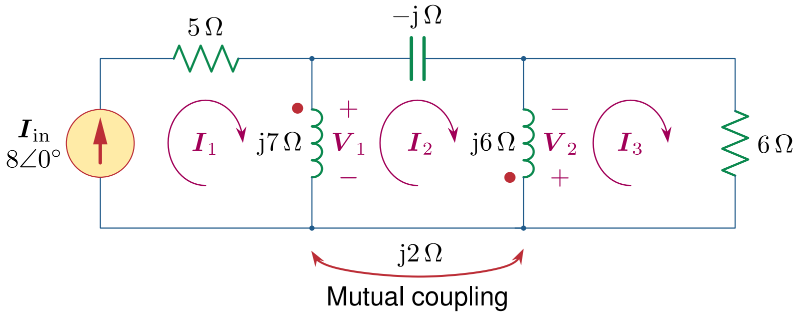


a) Calculate the equivalent resistance R_{eq}. (1 point)

b) Calculate the circuit quantities $\boldsymbol{I}_1$, $\boldsymbol{I}_2$, $\boldsymbol{I}_{\mathrm{L}}$ and $\boldsymbol{V}_{\mathrm{o}}$. (2 points)

c) Calculate the power dissipated in the $400\,\Omega$ resistor. (1 point)

Now, consider the circuit with magnetically coupled inductances in the figure below:



d) Redraw the circuit. Include the controlled voltage sources corresponding to the induced voltages, by also specifying their polarity and value. (2 points)

e) Express the phasor voltages $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ in terms of the phasor currents $\boldsymbol{I}_1$, $\boldsymbol{I}_2$ and $\boldsymbol{I}_3$. (2 points)

f) Give the mesh equations for $\boldsymbol{I}_2$ and $\boldsymbol{I}_3$ and determine the values of both $\boldsymbol{I}_2$ and $\boldsymbol{I}_3$. (2 points)
(*Hint:* Please note that $\boldsymbol{I}_1$ is already given!)

Indicate the measure units for all calculated quantities. Show all steps in your reasoning and never give numerical results without justification.

Solution

Sub-point (a)

$$R_{eq} = 4 + 48 \;//\; 400 \cdot \left(\frac{N_1}{N_2}\right)^2 = 4 + 12 = 16\Omega$$

Sub-point (b)

$$\boldsymbol{I_1} = \frac{16\angle 0°}{4 + 12} = 1\angle 0°\ A$$

$$\boldsymbol{I_2} = \frac{16\angle 0°}{4 + 12} \cdot \frac{48}{48 + 16} = 0{,}75\angle 0°\ A$$

$$\boldsymbol{I_L} = -\boldsymbol{I_2}\left(\frac{N_1}{N_2}\right) = -\frac{0{,}75\angle 0°}{5} = 0{,}15\ \angle 180°\ A$$

$$\boldsymbol{V_1} = \frac{12}{12 + 4} \cdot 16\angle 0° = 12\angle\ 0°\ V$$

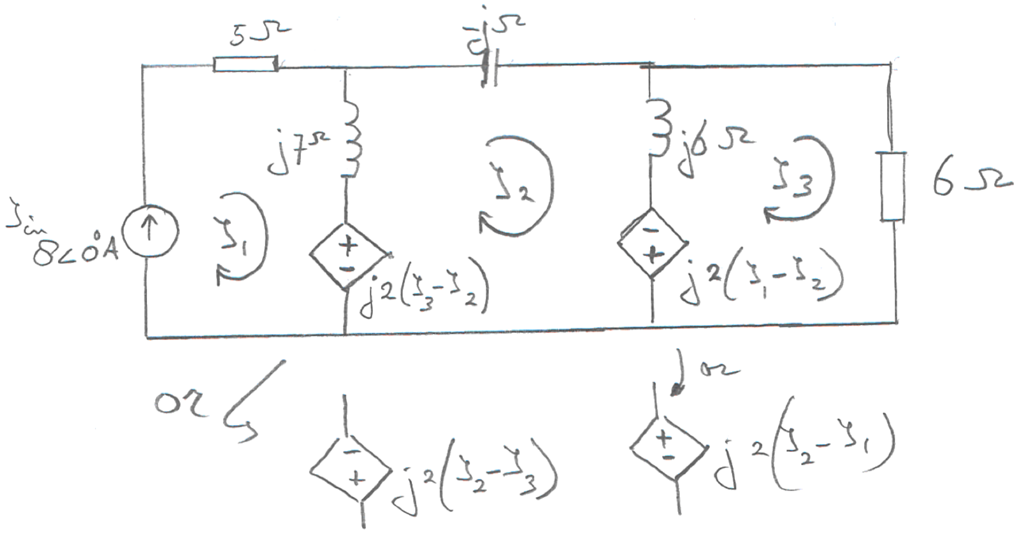
$$\boldsymbol{V_0} = -\left(\frac{N_2}{N_1}\right) \cdot \boldsymbol{V_1} = -5 \cdot \boldsymbol{V_1} = -5 \cdot 12\angle\ 0° = 60\ \angle\ 180°\ \ V$$

Sub-point (c)

$$P_{400} = \frac{V_{RmS}^2}{400} = I_{RmS}^2 . 400 = \frac{V_p^2}{2 \cdot 400} = 4{,}5\ W \quad (remember\colon V_{RMS} = \frac{V_p}{\sqrt{2}})$$

Note that the reference to the RMS was added for completeness, but was not requested by the exam exercise.

Sub-point (d)



Sub-point (e)

$$V_1 = \text{j}2(I_3 - I_2) + \text{j}7(I_1 - I_2)$$

$$V_2 = \text{j}6(I_3 - I_2) + \text{j}2(I_1 - I_2)$$

Sub-point (f)

Mesh I_2 : $j7(I_2 - I_1) - j2(I_3 - I_2) - jI_2 + j6(I_2 - I_3) - j2(I_1 - I_2) = 0$

$j16I_2 - j8I_3 = j9I_1$, given $I_1 = 8$, results in $\mathbf{2I_2 - I_3 = 9}$

Mesh I_3 : $6I_3 + j2(I_1 - I_2) + j6(I_3 - I_2) = 0$

$-j8I_2 + (j6+6)I_3 = -j2I_1$ with $I_1 = 8$ results in $\mathbf{-j4I_2 + (j3+3)I_3 = -j8}$

To solve the equations we substitute $2I_2 = I_3 + 9$. This results in $(j+3)I_3 = j10$ ➔ $\mathbf{I_3 = 3j+1}$ ➔ $\mathbf{I_2 = 1{,}5j+5}$

or : $\mathbf{I_3 = 3{,}16 \angle 16{,}7°}$ and $\mathbf{I_2 = 5{,}22 \angle 71{,}565°}$

- Take a new double-sheet -

Exercise 2

Determine the Laplace transform of the following functions:

a) $f(t) = \left(3t^2 + 2t + 1\right) e^{-5t}\, u(t)$ (2 points)

b) $g(t) = e^{-2t} \cos(t-4)\, u(t-4)$ (3 points)

Now, find the inverse Laplace transform of the following *s*-domain transfer functions:

c) $F(s) = \dfrac{s+4}{(s+3)s}$ (2 points)

d) $G(s) = \dfrac{s+5}{s^2 + 2s + 17}$ (3 points)

Show all steps in your reasoning and never give numerical results without justification.

Solution

Sub-point (a)

$$(3t^2+2t+1)e^{-5t}u(t)$$

$$(3t^2e^{-5t}+2te^{-5t}+e^{-5t})u(t)$$

$$3\cdot\frac{2}{(s+5)^3} \qquad 2\cdot\frac{1}{(s+5)^2} \qquad \frac{1}{s+5}$$

$$\rightarrow \quad F(s)=\frac{6}{(s+5)^3}+\frac{2}{(s+5)^2}+\frac{1}{s+5}$$

Sub-point (b)

$$\cos(t-4)\,u(t-4)\,e^{-2t}$$

IF $\mathcal{L}[\cos(t)] \rightarrow \frac{s}{s^2+1}$

THEN $\mathcal{L}[\cos(t-4)u(t-4)] \rightarrow \frac{s}{s^2+1}\cdot e^{-4s}$ TIME SHIFT PROPERTY

THEN $\mathcal{L}[e^{-2t}\cos(t-4)u(t-4)] \rightarrow \frac{s+2}{(s+2)^2+1}\,e^{-4(s+2)}$ FREQUENCY SHIFT PROPERTY

$$\rightarrow F(s)=e^{-4(s+2)}\cdot\frac{s+2}{(s+2)^2+1}$$

Sub-point (c)

$$\frac{s+4}{s(s+3)}=\frac{A}{s}+\frac{B}{s+3} \rightarrow \frac{4/3}{s}+\frac{-1/3}{s+3}$$

$$A=sF(s)\Big|_{s=0}=\frac{s+4}{(s+3)}=\frac{4}{3}$$

$$B=(s+3)F(s)\Big|_{s=-3}=\frac{(s+4)}{s}=\frac{1}{-3}=-\frac{1}{3}$$

$$\frac{4}{3}u(t)-\frac{1}{3}e^{-3t}u(t)$$

Sub-point (d)

$$F(s)=\frac{5+s}{17+2s+s^2}=\frac{s+5}{(s+1)^2+4^2}=\frac{s+1}{(s+1)^2+4^2}+\frac{4}{(s+1)^2+4^2}$$

$s^2+1+2s+16$

$\frac{s+1}{(s+1)^2+4^2} \rightarrow e^{-t}\cos 4t$

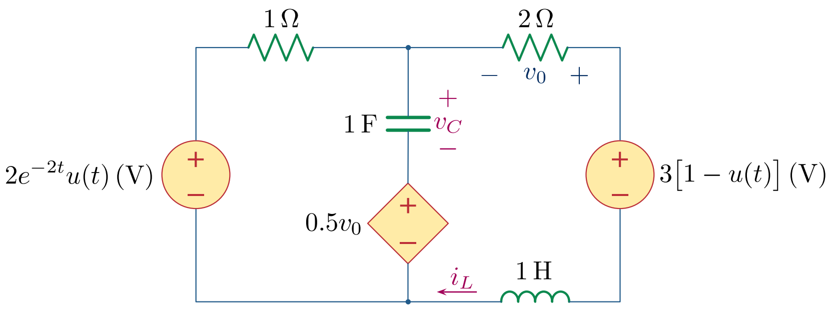
$\frac{4}{(s+1)^2+4^2} \rightarrow e^{-t}\sin 4t$

$$f(t)=e^{-t}(\cos 4t+\sin 4t)\,u(t)$$

- Take a new double-sheet -

Exercise 3

Consider the circuit in the figure below:



a) Calculate $v_C(0^+)$ and $i_L(0^+)$. (2 points)

b) Redraw the circuit in the Laplace-domain. (2 points)

c) Calculate the Laplace-domain current $\boldsymbol{I}_L(s)$. (4 points)
(*Hint:* Try applying mesh analysis.)

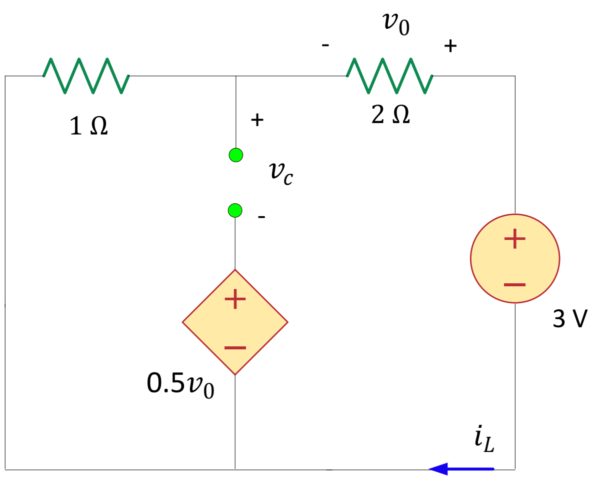
d) Calculate the time-domain current $i_L(t)$ for $t > 0$. (2 points)

Indicate the measure units for all calculated quantities. Show all steps in your reasoning and never give numerical results without justification.

Solution

Sub-point (a)

For t<0 the capacitor is open circuit and inductor is short circuit.



3 volts is across the 2 and 1 ohm resistors, therefore $v_0 = 2\ V$ and $i_L = -\frac{3}{3} = -1\ A$

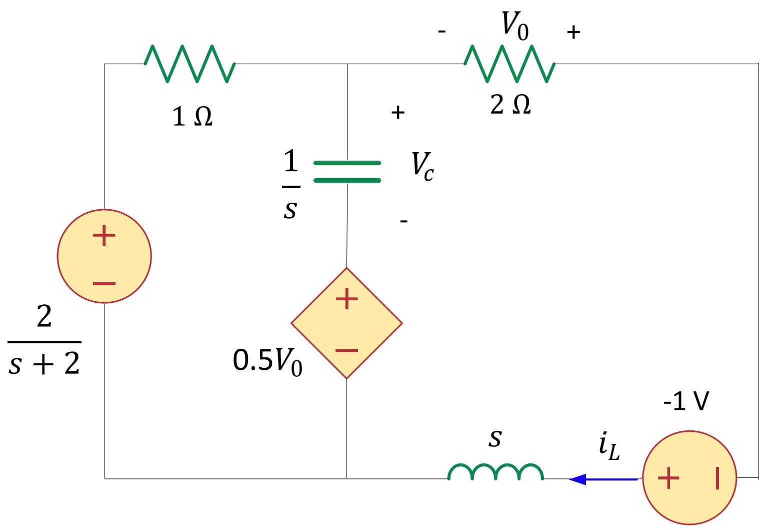
$$\rightarrow i_L(0^-) = -1\ A \rightarrow i_L(0^+) = -1\ A$$

KVL in the right loop at $t = 0^-$: $-0.5v_0 - v_c - v_0 + 3 = 0$

$$-1 - v_c - 2 + 3 = 0 \rightarrow v_c(0^-) = 0 \rightarrow v_c(0^+) = 0\ V$$

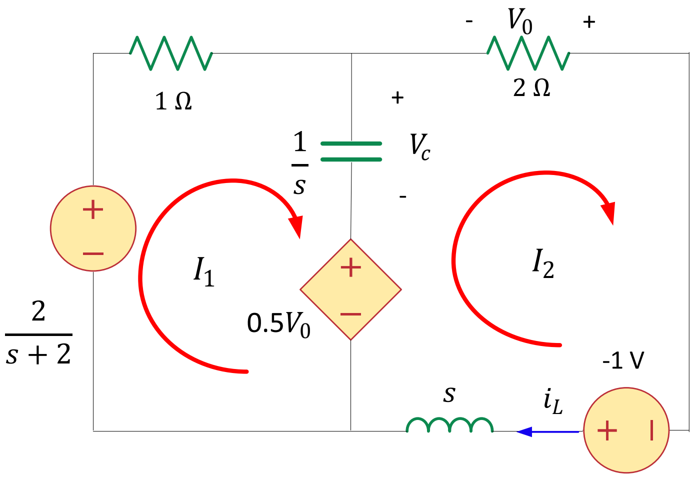
Sub-point (b)

For t>0 and by using the initial values, the Laplace-domain circuit becomes:



Sub-point (c)

By using the schematic shown at sub-point b., we write KVL on the left mesh and on the right mesh:



KVL (1):

$$-\frac{2}{s+2}+I_1+\frac{1}{s}(I_1-I_2)+0.5V_o=0$$

KVL (2):

$$-(-1)+sI_2-0.5V_o+\frac{1}{s}(I_2-I_1)-V_o=0$$

Using ohm law:

$$V_0=-2I_2$$

The two equations can be simplified to:

$$-\frac{2}{s+2}+I_1+\frac{1}{S}(I_1-I_2)-I_2=0$$

$$1+sI_2+\frac{1}{S}(I_2-I_1)+3I_2=0$$

Multiplying the second equation by s and after rearrangement:

$$I_1=s+s^2I_2+I_2+3sI_2$$

Replacing I_1 in the first equation:

$$-\frac{2}{s+2}+s+s^2I_2+I_2+3sI_2+\frac{1}{S}(s+s^2I_2+I_2+3sI_2-I_2)-I_2=0$$

$$-\frac{2}{s+2}+s+s^2I_2+3sI_2+(1+sI_2+3I_2)=0$$

$$s^2I_2+4sI_2+3I_2=-1-s+\frac{2}{s+2}$$

$$(s^2+4s+3)I_2=\frac{-(1+s)(s+2)+2}{s+2}$$

$$(s^2+4s+3)I_2=\frac{-s(s+3)}{s+2}$$

$$(s+3)(s+1)I_2=\frac{-s(s+3)}{s+2}$$

$$I_2 = I_{\mathrm{L}} = \frac{-s}{(s+2)(s+1)}$$

Sub-point (d)

By starting from the expression established at the previous sub-point, it can be successively inferred that:

$$I_2 = \frac{A}{(s+2)} + \frac{B}{(s+1)}$$

$$A = \frac{-(-2)}{(-2+1)} = -2$$

$$B = \frac{-(-1)}{(-1+2)} = 1$$

$$I_2 = I_{\mathrm{L}} = \frac{-2}{(s+2)} + \frac{1}{(s+1)}$$

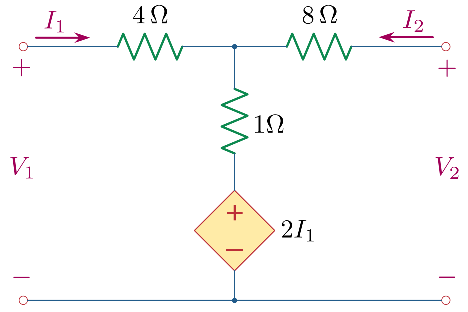
By now using the table of Laplace pairs and the transform's proprieties it follows that:

$$i_L(t) = (-2e^{-2t} + e^{-t})u(t) \text{ (A)}$$

- Take a new double-sheet -

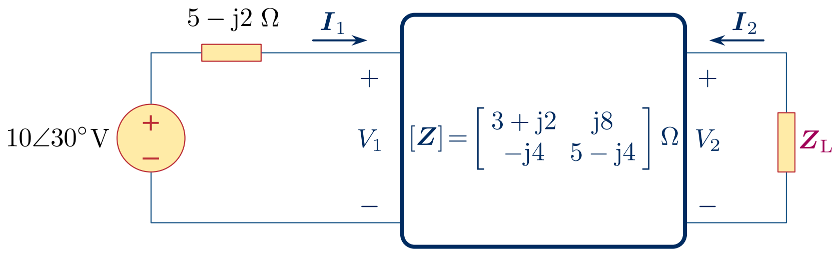
Exercise 4

Consider the circuit in the figure below:



a) Calculate the $\boldsymbol{Z}$–parameters z_{11}, z_{12}, z_{21} and z_{22}. (5 points)

Now consider the new circuit in the figure below:



b) Redraw the circuit with all independent sources pasivised (set to zero) and the load replaced by a current source with the value $\boldsymbol{I}_{test}$. (1 point)

c) Using the new circuit, find the Thévenin impedance seen at the load's terminals. (3 points)

d) Calculate $\boldsymbol{Z}_L$ for maximum power transfer. (1 point)

Indicate the measure units for all calculated quantities. Show all steps in your reasoning and never give numerical results without justification.

Solution

Sub-point (a)

KVL right loop:

$$-V_1 + 4I_1 + 1(I_1 + I_2) + 2I_1 = 0$$

$$Z_{11} = \frac{V_1}{I_1}\bigg|_{I_2=0} \Rightarrow V_1 = (4+1+2)I_1 \Rightarrow Z_{11} = 7\,\Omega$$

$$Z_{12} = \frac{V_1}{I_2}\bigg|_{I_1=0} \Rightarrow V_1 = 1I_2 \Rightarrow Z_{12} = 1\,\Omega$$

KVL left loop:

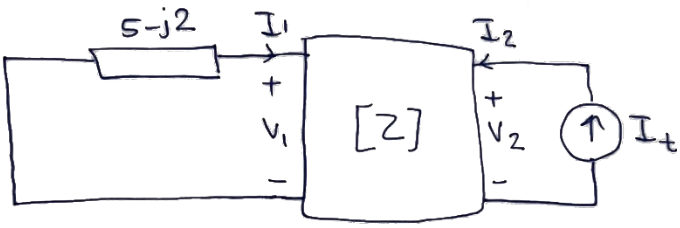
$$-V_2 + 8I_2 + 1(I_1 + I_2) + 2I_1 = 0$$

$$Z_{21} = \frac{V_2}{I_1}\bigg|_{I_2=0} \Rightarrow V_2 = 3I_1 \Rightarrow Z_{21} = 3\,\Omega$$

$$Z_{22} = \frac{V_2}{I_2}\bigg|_{I_1=0} \Rightarrow V_2 = (8+1)I_2 \Rightarrow Z_{22} = 9\,\Omega$$

Sub-point (b)

The circuit obtained by passivising (setting to zero) the voltage source and by replacing the load by a test current source with the value $\boldsymbol{I}_{\text{test}}$ is:



Sub-point (c)

Calculating the Thévenin impedance is carried out by employing the circuit given at sub-point b. The fact that $\mathbf{z}_{12} \neq \mathbf{z}_{21}$ implies that the circuit inside two-port is non-reciprocal which, in turn, implies that, quite likely, the circuit contains controlled sources.

$$[Z] \Rightarrow \begin{cases} V_1 = (3+j2)I_1 + j8I_2 & \text{I} \\ V_2 = -j4I_1 + (5-j4)I_2 & \text{II} \end{cases}$$

$$I_2 = I_t = 1\text{A} \quad \text{III} \qquad , \qquad V_1 = -(5-j2)I_1 \quad \text{IV}$$

$$\text{I} \ \& \ \text{IV} : \ -(5-j2)I_1 = (3+j2)I_1 + j8 \Rightarrow I_1 = -j$$

$$\text{II} : \ V_2 = -4 + (5-j4) = 1 - j4$$

$$Z_{th} = V_2 / I_2 = 1 - j4 \ \Omega$$

Sub-point (d)

By conjugate matching it follows that $\mathbf{Z}_L = \mathbf{Z}_{Th}^* = 1 + j4\ (\Omega)$.