End-of-term Exam

EE1C21 "Linear Circuits B"

Place: Drebbelweg Exam Hall 2
Date: 03-02-2023
Time: 9:00 – 11:00

- This exam consists of 4 exercises.
- Each exercise accounts for **10 points**; the total number of points to be obtained is **40**. The exam grade is obtained by dividing the total number of points by 4, rescaling linearly the result to the 1-10 scale and rounding off to 1 decimal.
- **Each exercise must be solved on a separate double-sheet.** Writing more solutions on the same sheet may result in only one of the solutions being graded!
- Indicate your name and study number on **each** submitted sheet. **You must hand in (blank) signed sheets even for the exercises that you do not handle.**
- Students benefitting of the "Extra Time" (ET) rule are entitled to a 20 minutes extension of their exam provided they produce the relevant supporting document.
- Should any question not be completely clear, you are allowed to ask the instructors in the exam hall; the answer will be confined to rephrasing the text of the exercise such that to make it more intelligible.
- Should a part of an exercise depend on a previous result, mistakes made at a previous step will only be penalised once.
- Give your solution as completely as possible and never state numerical results without indicating how you derived them. **Simply stating numerical results will yield no points.**
- **When requested, fill in the measure units for all calculated quantities.** This holds for intermediate results but definitely for the final ones.
- Write clearly and avoid messy solutions. Should errors occur in your solution, cross the erroneous part out and give clear indications on where the correct solution resumes.
- For this exam you are allowed to use:
    i. a simple calculator – programmable and graphic calculators are explicitly prohibited;
    ii. a handwritten, double-sided A4 sheet with formulas.
- The text of this exam is offered only in English. Inasmuch as possible, instructors will assist you with the Dutch translation of formulations that you may have difficulties to understand.

The Linear Circuits team wishes you a lot of success!

Laplace transform pairs.*

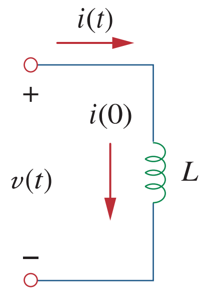
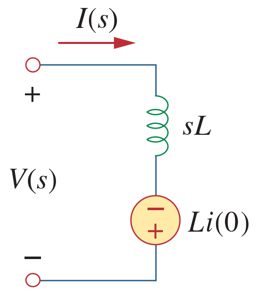
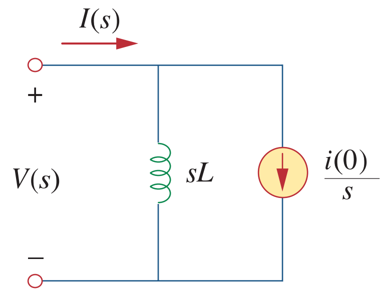
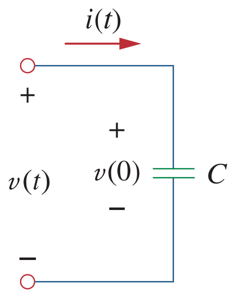
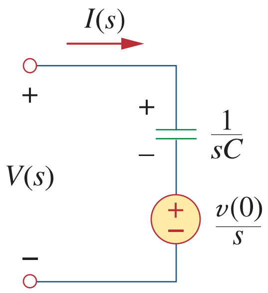
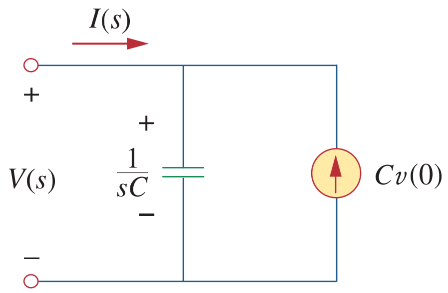
$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t + \theta)$	$\frac{s \sin\theta + \omega \cos\theta}{s^2+\omega^2}$
$\cos(\omega t + \theta)$	$\frac{s \cos\theta - \omega \sin\theta}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

*Defined for $t \geq 0$; $f(t) = 0$, for $t < 0$.

Properties of the Laplace transform.

Property	$f(t)$	$F(s)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Scaling	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
Time shift	$f(t-a)u(t-a)$	$e^{-as}F(s)$
Frequency shift	$e^{-at}f(t)$	$F(s+a)$
Time differentiation	$\frac{df}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2f}{dt^2}$	$s^2F(s) - sf(0^-) - f'(0^-)$
	$\frac{d^3f}{dt^3}$	$s^3F(s) - s^2f(0^-) - sf'(0^-) - f''(0^-)$
	$\frac{d^nf}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \cdots - f^{(n-1)}(0^-)$
Time integration	$\int_0^t f(x)\,dx$	$\frac{1}{s}F(s)$
Frequency differentiation	$tf(t)$	$-\frac{d}{ds}F(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(s)\,ds$
Time periodicity	$f(t) = f(t+nT)$	$\frac{F_1(s)}{1-e^{-sT}}$
Initial value	$f(0)$	$\lim_{s\to\infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s\to 0} sF(s)$
Convolution	$f_1(t) * f_2(t)$	$F_1(s)F_2(s)$

Laplace-domain equivalent circuits for inductances and capacitances

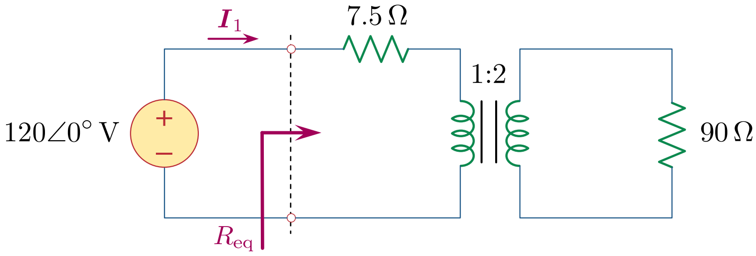
Time-domain circuit	Thévenin-type equivalent	Norton-type equivalent
		
		

Initial-conditions voltage/current values: $v(0) = v(0^-)$ and $i(0) = i(0^-)$.

- Take a new double-sheet -

Exercise 1

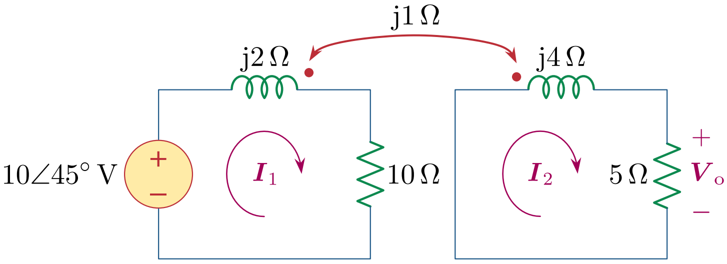
Consider the circuit with an ideal transformer in the figure below:



a) Calculate the equivalent resistance R_{eq}. (1 point)

b) Calculate the current $\boldsymbol{I}_1$. (1 point)

Now consider the circuit with magnetically coupled inductors in the figure below:

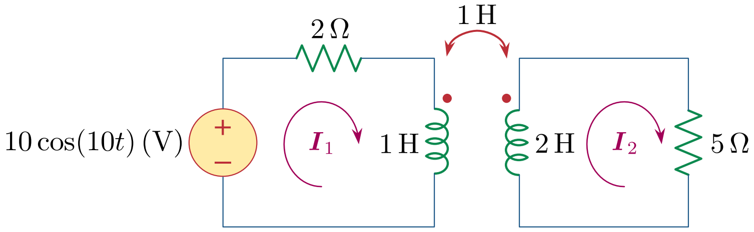


c) Calculate the coupling coefficient k assuming $\omega = 1$ rad/s. (1 point)

d) Redraw the equivalent circuit where the magnetically coupled inductors are replaced by normal inductors and dependent voltage sources. (1 point)
Hint: Remember the dot convention for the polarity of the voltage sources.

e) Write the mesh equations for the currents $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ and then calculate the value of $\boldsymbol{V}_o$ using the value of $\boldsymbol{I}_2$. (2 points)

Now consider the new circuit with magnetically coupled inductors in the figure below:



f) Redraw the equivalent circuit where the magnetically coupled inductors are replaced by normal inductors and dependent voltage sources. (1 point)

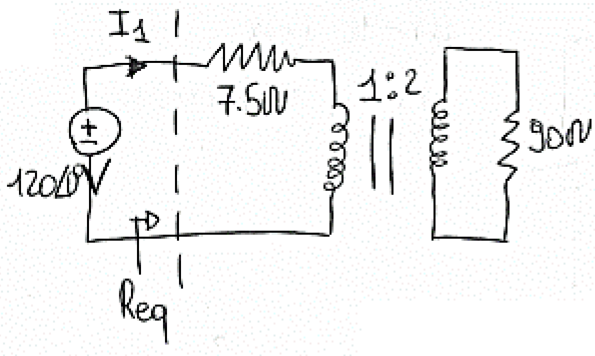
g) Write the mesh equations for the currents $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ and then calculate their expression in the time domain. (2 points)

h) Calculate the energy stored in the magnetically coupled inductors at time $t = 1$s. (1 point)

Indicate the measure units for all calculated quantities. Show all steps in your reasoning and never give numerical results without justification.

Solution

Sub-point (a)



$$R_{eq} = Z_{REF} + 7.5$$

$$Z_{REF} = \frac{90}{\left(\frac{2}{1}\right)^2} = 22.5\Omega$$

$$\text{so } \underline{R_{eq} = 30\Omega}$$

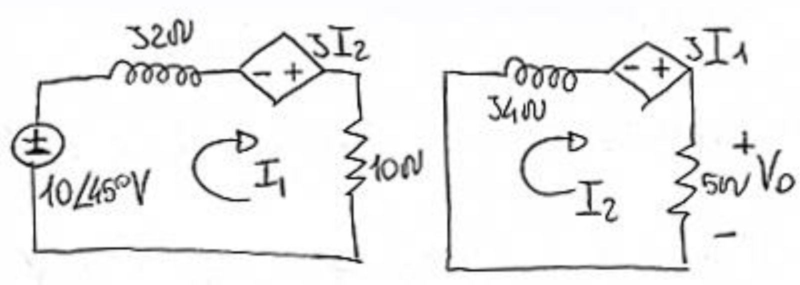
Sub-point (b)

$$I_1 = V/R_{eq} = 120/30 = \underline{4\text{ A}}$$

Sub-point (c)

$$K = \frac{M}{\sqrt{L_1 L_2}} = \frac{1}{\sqrt{2 \cdot 4}} = \underline{0.35}$$

Sub-point (d)



Sub-point (e)

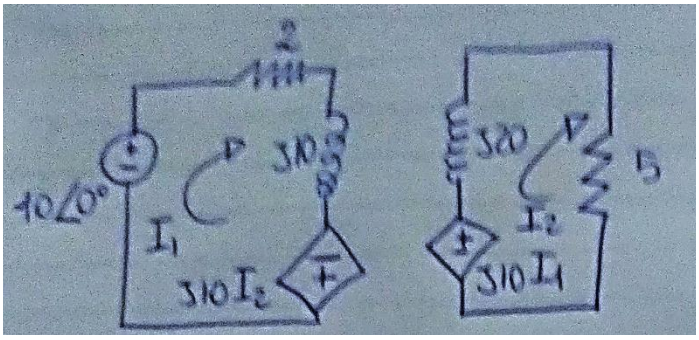
$$\begin{cases} 10\angle 45^\circ + jI_2 = j2I_1 + 10I_1 \\ j4I_2 + 5I_2 = jI_1 \end{cases} \quad \begin{cases} 10\angle 45^\circ = -jI_2 + (j2+10)(4I_2 - j5I_2) \\ I_1 = 4I_2 - j5I_2 \end{cases}$$

$$\begin{cases} 10\angle 45^\circ = -jI_2 + j8I_2 + 10I_2 + 40I_2 - j50I_2 \\ / \end{cases} \quad \begin{cases} 10\angle 45^\circ = 50I_2 - j43I_2 \\ / \end{cases}$$

$$I_2 = \frac{10\angle 45^\circ}{50 - j43} = \frac{7.1 + j7.1}{50 - j43} = 0.01 + j0.15\ A$$

$$V_o = 5 \cdot I_2 = \frac{35.5 + j35.5}{50 - j43} = 0.06 + j0.76 = 0.76\angle 85.7^\circ\ V$$

Sub-point (f)



This is the circuit in the phasor domain

Sub-point (g)

$$\begin{cases} 10 + j10I_2 = 2I_1 + j10I_1 \\ j10I_1 = j20I_2 + 5I_2 \end{cases} \quad \begin{cases} 10 = -j10I_2 + (2+j10)(2I_2 - j0.5I_2) \\ I_1 = \frac{5I_2 + j20I_2}{j10} = 2I_2 - j0.5I_2 \end{cases}$$

$$\begin{cases} 10 = -j10I_2 + 4I_2 - jI_2 + 20jI_2 + 5I_2 \\ / \end{cases} \quad \begin{cases} 10 = 9I_2 + 9jI_2 \\ / \end{cases} \quad \begin{cases} I_2 = \frac{10}{9+9j} \\ / \end{cases}$$

$$\begin{cases} I_2 = \frac{10}{9+9j} = 0.56 - 0.56j = 0.79\angle -45^\circ \rightarrow \underline{0.79\cos(10t - 45^\circ)\ A} \\ I_1 = \frac{10(2 - j0.5)}{9+9j} = 0.83 - 1.39j = 1.62\angle -59^\circ \rightarrow \underline{1.62\cos(10t - 59^\circ)\ A} \end{cases}$$

Time-domain expressions of the currents

Sub-point (h)

Energy analysis

$$I_1(t)\big|_{t=1} = 1.62\cos(10-59^\circ) = -1.46\ A$$

$$I_2(t)\big|_{t=1} = 0.79\cos(10-45^\circ) = -0.77\ A$$

$$W = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 - M I_1 I_2 =$$

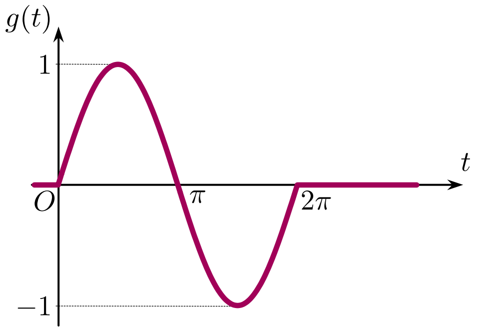
$$= \frac{1}{2}\cdot 1\,(-1.46)^2 + \frac{1}{2}\cdot 2\,(-0.77)^2 - (-0.77)(-1.46) = \boxed{0.535\ J}$$

- Take a new double-sheet -

Exercise 2

a) Calculate the Laplace transform $F(s)$ of the function $f(t) = e^{-2t}u(t-\tau)$. (2 points)

b) Calculate the Laplace transform $G(s)$ of the function shown in the plot below. (2 points)



c) Calculate the inverse Laplace transform $f(t)$ of the s-domain transfer function $F(s) = \dfrac{s+1}{s+3}$. Use the final value theorem to verify your answer.

(2 points)

d) Calculate the inverse Laplace transform $g(t)$ of the s-domain transfer function $G(s) = \dfrac{3s+9}{(s^2+s+1)(s+2)}$. (4 points)

Show all steps in your reasoning and never give numerical results without justification.

Solution

Sub-point (a)

$$e^{-2t}u(t-\tau) = e^{-2(t-\tau)-2\tau}u(t-\tau) = e^{-2\tau} \times e^{-2(t-\tau)}u(t-\tau)$$

$$\mathcal{L}\left(e^{-2t}u(t-\tau)\right) = e^{-2\tau}\frac{e^{-s\tau}}{s+2} = \frac{e^{-(s+2)\tau}}{s+2}$$

Sub-point (b)

$$f(t) = \sin(t)\left(u(t) - u(t-2\pi)\right) = \sin(t)\,u(t) - \sin(t)\,u(t-2\pi) =$$

$$= \sin(t)\,u(t) - \sin\left((t-2\pi)+2\pi\right)u(t-2\pi)$$

$$= \sin(t)\,u(t) - \sin(t-2\pi)\,u(t-2\pi)$$

$$\mathcal{L}(f(t)) = \frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1} = \frac{1-e^{-2\pi s}}{s^2+1}$$

Sub-point (c)

$$F(s) = \frac{s+1}{s+3} = \frac{s+3-2}{s+3} = 1 - \frac{2}{s+3}$$

$$f(t) = \delta(t) - 2e^{-3t}u(t)$$

Final theorem:

$$\lim_{s\to 0} sF(s) = s\frac{s+1}{s+3} = 0$$

$$g(\infty) = \delta(\infty) - 2e^{-3\infty} = 0$$

Sub-point (d)

$$G(s) = \frac{3s+9}{(s^2+s+1)(s+2)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+s+1}$$

$$A = (s+2)F(s)\Big|_{s\,=\,-2} = 1$$

$$\frac{1}{s+2} + \frac{Bs+C}{s^2+s+1} = \frac{s^2+s+1+Bs^2+2Bs+Cs+2C}{(s^2+s+1)(s+2)}$$

Therefore: $B = -1, C = 4$

$$G(s) = \frac{1}{s+2} - \frac{s-4}{s^2+s+1}$$

$$G(s) = \frac{1}{s+2} - \frac{s-4}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$G(s) = \frac{1}{s+2} - \frac{s + \frac{1}{2} - 4.5}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

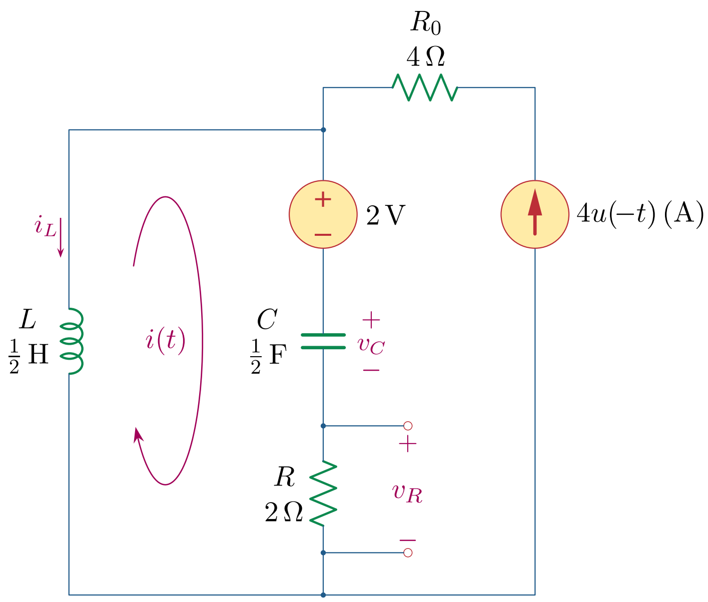
$$G(s) = \frac{1}{s+2} - \frac{s + \frac{1}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{9}{\sqrt{3}} \frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$g(t) = \left(e^{-2t} - e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2} t\right) + \frac{9}{\sqrt{3}} e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2} t\right)\right) u(t)$$

- Take a new double-sheet -

Exercise 3

Consider the circuit in the figure below:



a) Calculate $v_C(0-)$ and $i_L(0-)$ in the time-domain. (1 point)

b) Redraw the circuit in the s-domain, by also accounting for the initial states and their values. (3 points)

c) Obtain the expression of the current $\boldsymbol{I}(s)$ in the s-domain that corresponds to the mesh current $i(t)$. (3 points)

d) What type of damping we have in this circuit? (1 point)

e) Calculate the inverse Laplace transform $i(t)$ and obtain the resistor voltage $v_R(t)$ in the time-domain for $t > 0$. (2 points)

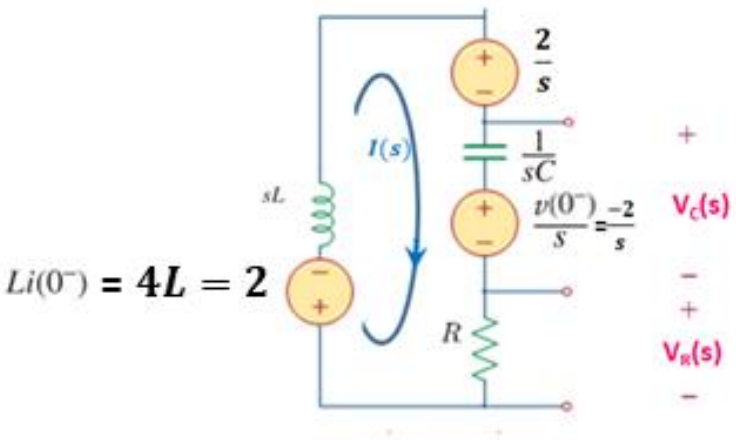
Indicate the measure units for all calculated quantities. Show all steps in your reasoning and never give numerical results without justification.

Solution

Sub-point (a)

$v_C(0-) = -2$ V and $i_L(0-) = 4$ A

Sub-point (b)



Sub-point (c)

Mesh for I:

$$\frac{2}{s} + \frac{I}{sC} + \frac{-2}{s} + IR + Li_L(0) + sLI = 0$$

$$I\frac{L}{s}\left(s^2 + \frac{R}{L}S + \frac{1}{LC}\right) = -Li_L(0)$$

$$I(s) = \frac{-Li_L(0)}{\frac{L}{s}\left(s^2 + \frac{R}{L}S + \frac{1}{LC}\right)} = \frac{-si_L(0)}{s^2 + \frac{R}{L}S + \frac{1}{LC}} = \frac{-4s}{s^2 + 4S + 4} = \frac{-4s}{(s+2)^2} = \frac{A}{(s+2)^2} + \frac{B}{(s+2)}$$

Sub-point (d)

Characteristic equation: $s^2 + 4S + 4 = 0$ $\sqrt{(\alpha^2 - \omega_0{}^2)} = \sqrt{(2^2 - 4} = 0$ → Critically damped

Sub-point (e)

Solving for A and B gives: A = 8 and B = -4: $I(s) = \frac{8}{(s+2)^2} + \frac{-4}{(s+2)}$

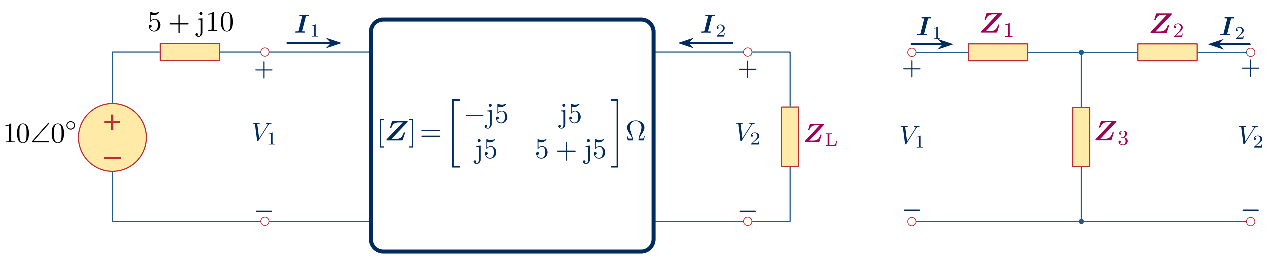
Applying the inverse Laplace transform: $\boldsymbol{i(t) = (-4 + 8t)e^{-2t}}$ $(\boldsymbol{A})$ for $\boldsymbol{t > 0}$

So: $\boldsymbol{v_R(t) = (-8 + 16t)e^{-2t}}$ $(\boldsymbol{V})$ for $\boldsymbol{t > 0}$.

- Take a new double-sheet -

Exercise 4

Consider the circuit at the left-hand side, in the figure below:

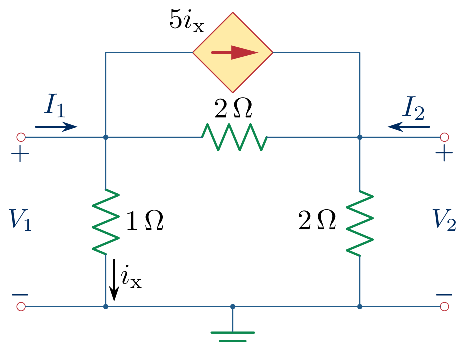


a) Calculate the impedances $\boldsymbol{Z}_1$, $\boldsymbol{Z}_2$ and $\boldsymbol{Z}_3$ of its T equivalent circuit shown at the right-hand side of the figure. (1 points)

b) Calculate the Thévenin impedance $\boldsymbol{Z}_{\mathrm{Th}}$ and voltage $\boldsymbol{V}_{\mathrm{Th}}$ of the Thévenin equivalent seen at the terminals of the load $\boldsymbol{Z}_{\mathrm{L}}$. (3 points, 1.5 points each)
Hint: You may consider replacing the two-port by its T equivalent circuit derived at sub-point (a).

c) Calculate the maximum power transferred to the load $\boldsymbol{Z}_{\mathrm{L}}$, under conjugate matching conditions. (1 point)

Now consider the new circuit in the figure below:



d) Calculate the $\boldsymbol{Y}$–parameters $\boldsymbol{y}_{11}$, $\boldsymbol{y}_{12}$, $\boldsymbol{y}_{21}$ and $\boldsymbol{y}_{22}$. (5 points)

Indicate the measure units for all calculated quantities. Show all steps in your reasoning and never give numerical results without justification.

Solution

Sub-point (a)

By applying the standard relations, the impedances in the T equivalent circuit are

$$\boldsymbol{Z}_1 = \boldsymbol{z}_{11} - \boldsymbol{z}_{12} = -\mathrm{j}5 - \mathrm{j}5 = -\mathrm{j}10$$

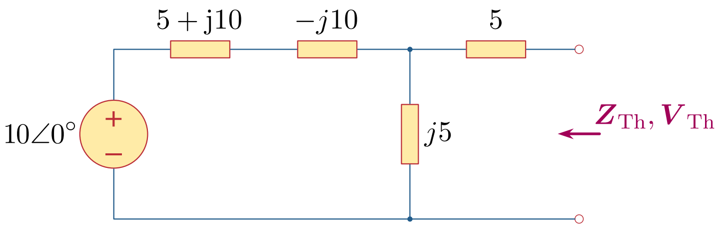
$$\boldsymbol{Z}_2 = \boldsymbol{z}_{22} - \boldsymbol{z}_{12} = 5 + \mathrm{j}5 - \mathrm{j}5 = 5$$

$$\boldsymbol{Z}_3 = \boldsymbol{z}_{12} = \mathrm{j}5$$

respectively. Note that, since all impedances are complex, no measure unit was filled in. However, filling in an ohm is perfectly acceptable.

Sub-point (b)

By replacing the two-port by its T equivalent circuit, the circuit at the terminals of the load $\boldsymbol{Z}_\mathrm{L}$ becomes the one in the figure below.



For calculating the Thévenin impedance $\boldsymbol{Z}_\mathrm{Th}$, we turn off (passivize) the independent voltage source and observe that the two series impedances yield the equivalent impedance

$$\boldsymbol{Z}_\mathrm{eq} = (5 + \mathrm{j}10) + (-\mathrm{j}10) = 5$$

The remaining circuit consists of $\boldsymbol{Z}_\mathrm{eq}$ in parallel with the $j5$ impedance, and in series with the 5 impedance. It is then clear that the Thévenin impedance $\boldsymbol{Z}_\mathrm{Th}$ is

$$\boldsymbol{Z}_\mathrm{Th} = 5 + \frac{\mathrm{j}25}{5 + \mathrm{j}5} = \frac{5 + \mathrm{j}5 + \mathrm{j}5}{1 + \mathrm{j}} = \frac{5 + \mathrm{j}10}{1 + \mathrm{j}} = \frac{(5 + \mathrm{j}10)(1 - \mathrm{j})}{2} = \frac{15}{2} + \mathrm{j}\frac{5}{2}$$

As for the Thévenin voltage $\boldsymbol{V}_\mathrm{Th}$, it readily follows by dividing the 10 voltage source between $\boldsymbol{Z}_\mathrm{eq}$ and the $j5$ impedance

$$\boldsymbol{V}_\mathrm{Th} = 10\frac{\mathrm{j}5}{5 + \mathrm{j}5} = \frac{\mathrm{j}10}{1 + \mathrm{j}} = 5\mathrm{j}(1 - \mathrm{j}) = 5 + \mathrm{j}5$$

Note that the same observation concerning the measure units applies in this case.

Sub-point (c)

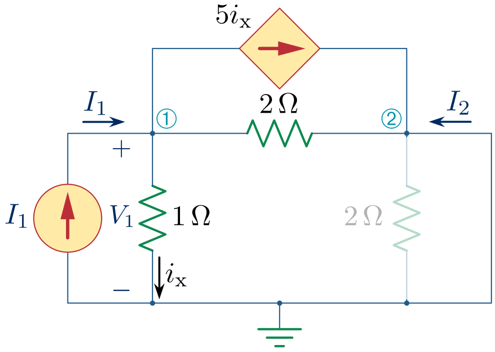
The maximum power transfer is obtained under complex conjugation matching, with the Thévenin resistance being the real part of the Thévenin impedance $\boldsymbol{Z}_\mathrm{Th}$. The maximum power is then

$$P_\mathrm{max} = \frac{|\boldsymbol{V}_\mathrm{Th}|^2}{8R_\mathrm{Th}} = \frac{|\boldsymbol{V}_\mathrm{Th}|^2}{8\,\mathrm{Re}(\boldsymbol{Z}_\mathrm{Th})} = \frac{|5 + \mathrm{j}5|^2}{8 \times 15/2} = \frac{5}{6} = 0.833\ (\mathrm{W})$$

Note that filling in the measure unit *is mandatory* in this case (the power always carries a measure unit since it is a real quantity), and failing to give the correct measure unit is penalised.

Sub-point (d)

Since the given two-port contains a dependent source, determining the $\boldsymbol{Y}$–parameters requires using test sources. For simplicity, current sources injecting either I_1 or I_2, with the needed orientation, will be used as test sources. Firstly, a test source is placed at the left port, with the right port being short-circuited → the resulting circuit is shown below



We now observe that the short circuit renders the 2Ω resistance redundant (no current will pass through it). Application of KCL at ① will yield

$$\frac{V_1}{1} + \frac{V_1}{2} + 5i_x = I_1 \longrightarrow \frac{V_1}{1} + \frac{V_1}{2} + 5\frac{V_1}{1} = I_1 \longrightarrow V_1 = \frac{2I_1}{13}\,(\mathrm{V})$$

while application of KCL at ② will yield

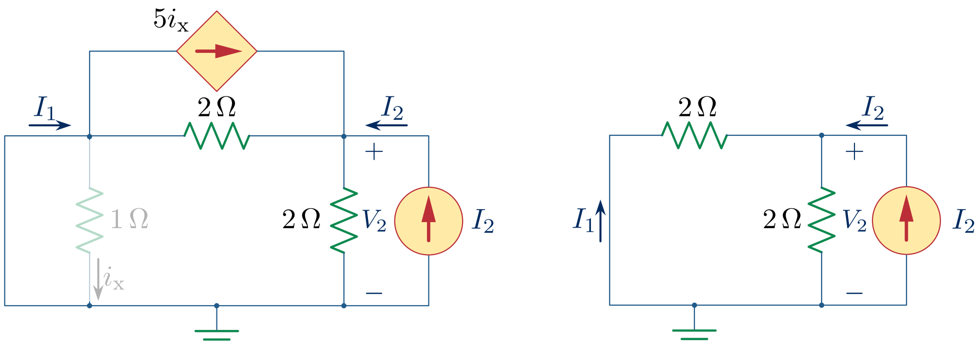
$$I_2 = -\frac{V_1}{2} - 5i_x = -\frac{V_1}{2} - 5\frac{V_1}{1} = -V_1\,\frac{11}{2}\,(\mathrm{A})$$

We can now obtain two $\boldsymbol{Y}$–parameters, namely

$$\boldsymbol{y}_{11} = \frac{I_1}{V_1} = \frac{13}{2}\,(\mathrm{S})$$

$$\boldsymbol{y}_{21} = \frac{I_2}{V_1} = -\frac{11}{2}\,(\mathrm{S})$$

Secondly, a test source is placed at the right port, with the left port being short-circuited → the resulting circuit is shown below – see subfigure (a).



a b

Similarly to the previous case, the short circuit renders the 1Ω resistance redundant (no current will pass through it) and the circuit simplifies to the one in the subfigure (b). Analysing this circuit is elementary: Ohm's law yields

$$V_2 = 1\, I_2\, (\mathrm{V})$$

while a current division yields

$$I_1 = -I_2 \frac{1}{2}\, (\mathrm{A})$$

We can now obtain the remaining two $\boldsymbol{Y}$–parameters, namely

$$\boldsymbol{y}_{22} = \frac{I_2}{V_2} = 1\, (\mathrm{S})$$

$$\boldsymbol{y}_{12} = \frac{I_1}{V_2} = -\frac{1}{2}\, (\mathrm{S})$$

In a compact form, the two-port's Y-matrix is

$$[\boldsymbol{Y}] = \begin{bmatrix} \frac{13}{2} & -\frac{1}{2} \\ -\frac{11}{2} & 1 \end{bmatrix} (\mathrm{S})$$

(this form was not requested in the exercise, it is only given here for completeness). Note that measure units were filled in throughout this sub-point since all calculated values are real (although all two-port parameters, the input voltages and currents included, are, in principle, complex quantities). Both filling in the correct measure unit or leaving the measure units out are acceptable.