Exam 2 - EE1M1 Calculus (30/01/2024 09:00 - 11:00)

Fill in your personal information and
write down your answers for the eight short-answer questions and
write down all your steps for the five open question and
hand in when finished.

You are allowed to use:

- Pen, pencils and scrap paper;
- A simple calculator;
- The formula sheet;

Short-answer questions

An explanation is not required for the short-answer questions. Only the answer matters. The maximum points per question is indicated in the margin.

Clearly write the answer in the box. You do not need to fully simplify your answers.

1. (2 +2 pt) Let $f(x,y) = 3x^3 - 2xy^2$.

   a. Compute the gradient ∇f of f.

   b. For which direction $\mathbf{u}$ does $D_{\mathbf{u}} f(1,2)$ reach its minimal value?

   A correct solution is: a. The gradient can be found by computing the partial derivatives of f. We have

   $$f_x = 9x^2 - 2y^2, f_y = -4xy$$

   It follows that $\nabla f = \langle 9x^2 - 2y^2, -4xy \rangle$.

   b. The minimal value of the directional derivative is reached in the direction of $-\nabla f(1,2)$. We compute $-\nabla f(1,2) = -\langle 1, -8 \rangle = \langle -1, 8 \rangle$. Any positive multiple of this vector is a correct answer, including the normalized version $\left\langle -\frac{1}{\sqrt{65}}, \frac{8}{\sqrt{65}} \right\rangle$.

2. (4 pt) Find all critical points of the function $f(x,y) = x^2 - 2x + 3y - y^3$ and classify them as local maxima, local minima or saddle points.

   A correct solution is: The critical points can be found by finding the points where both partial derivatives are 0. We compute

   $$f_x = 2x - 2, f_y = 3 - 3y^2.$$

   Setting $f_x = 0$ yields $x = 1$, while setting $f_y = 0$ yields $y = -1$ or $y = 1$. Hence, f has two critical points, $(1,-1)$ and $(1,1)$. We can classify these by using the second derivatives test. We first compute

   $$f_{xx} = 2, f_{xy} = f_{yx} = 0, f_{yy} = -6y.$$

   The discriminant can now be evaluated as $D = f_{xx} f_{yy} - f_{xy}^2 = -12y$. So we find $D(1,-1) = 12$ and $D(1,1) = -12$. Since $f_{xx}(1,-1) = 2 > 0$, f has a local minimum at $(1,-1)$, while f has a saddle point at $(1,1)$.

3. (4 pt) Reverse the order of integration for $\displaystyle\int_0^4 \int_{1-\frac{x}{2}}^1 f(x,y)\, dy\, dx$ and give the resulting integral.

   A correct solution is: We need to sketch the region of integration in order to switch the order of integration. The limits for y yield that $1 - \frac{x}{2} \leq x \leq 1$, so the region is bounded in between the lines $y = 1 - \frac{x}{2}$ and $y = 1$. These lines intersect at $(0,1)$. Since the limits for x yield that $0 \leq x \leq 4$, we can conclude that the region of integration is a triangle with vertices $(0,1)$, $(4,1)$ and $(4,-1)$. In order to describe this region as a type II (x-simple) region, we should first find the new limits for x. The triangle is bounded on the left by the line $y = 1 - \frac{x}{2}$, i.e. the line $x = 2 - 2y$, and on the right by the line $x = 4$. In addition, the allowed values for y range from -1 at one of the vertices to 1 at the line between the other two vertices. So we obtain

   $$\int_0^4 \int_{1-\frac{x}{2}}^1 f(x,y)\, dy\, dx = \int_{-1}^1 \int_{2-2y}^4 f(x,y)\, dx\, dy.$$

4. (4 pt) Is the vector field $\mathbf{F}(x, y, z) = \langle ze^{xz}, 1 + z, xe^{xz} + y\rangle$ conservative? If it is conservative, give a potential function.

A correct solution is: If ϕ were a potential for $\mathbf{F}$, then, since $\frac{\partial \phi}{\partial x} = ze^{xz}$, we should have $\phi = e^{xz} + h(y, z)$, where h is a function of only y and z. Since $\frac{\partial \phi}{\partial y} = \frac{\partial h}{\partial y}$, we should have $\frac{\partial h}{\partial y} = 1+z$. Hence, we must have that $h(y, z) = y+yz+g(z)$, where g is a function of only z. So we must have $\phi = e^{xz}+y+yz+g(z)$. Finally, we compute $\frac{\partial \phi}{\partial z} = xe^{xz}+y+\frac{dg}{dz}$, which should be equal to $xe^{xz} + y$. This is true whenever g is a constant. So $\mathbf{F}$ is conservative and $\phi = e^{xz} + y + yz + C$ is a potential function for any constant C.

5. (6 pt) Let $\mathcal{D}$ be the region in $\mathbb{R}^2$ bounded by the circle $(x - 2)^2 + y^2 = 4$, the x-axis and the line $y = x$. A charge density $q(x, y)$ is distributed over this region. Express the net total charge Q on $\mathcal{D}$ as a double integral in polar coordinates.

A correct solution is: We first describe the region $\mathcal{D}$ in polar coordinates. The lowest value of r occurs at the origin, where $r = 0$. The circle $(x - 2)^2 + y^2 = 4$ can be written as $x^2 - 4x + 4 + y^2 = 4$, i.e. $x^2 + y^2 = 4x$. In polar coordinates, this equation becomes $r^2 = 4r\cos(\theta)$, i.e. $r = 4\cos(\theta)$.

The circle is contained entirely in the right-half plane, i.e. the part of $\mathbb{R}^2$ with $x \geq 0$, so we only need to consider points with $x \geq 0$. The positive x-axis consists of all points which make angle 0 with the positive x-axis, so it can be described as the equation $\theta = 0$. Finally, the part of the line $y = x$ with $x \geq 0$ consists of all points which make angle $\frac{\pi}{4}$ with the positive x-axis, so it can be described as the equation $\theta = \frac{\pi}{4}$.

So $\mathcal{D} = \{(r\cos(\theta), r\sin(\theta) \,|\, 0 \leq r \leq 4\cos(\theta), 0 \leq \theta \leq \frac{\pi}{4}\}$. We conclude that

$$Q = \iint_{\mathcal{D}} \sigma(x, y)\, dA = \int_0^{\frac{\pi}{4}} \int_0^{4\cos(\theta)} \sigma(r\cos\theta, r\sin(\theta))r\, dr\, d\theta.$$

Here we inserted the polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$ into the integrand $\sigma(x, y)$ and we multiplied the entire integrand by the Jacobian r.

6. (6 pt) Let $\mathcal{E}$ be the solid region in $\mathbb{R}^3$ which is given by the part with $y \leq 0$ of the region in between the cone $z = \sqrt{3x^2 + 3y^2}$ and the plane $z = 2$. Express the integral $\iiint_{\mathcal{E}} \sqrt{x^2 + y^2 + z^2}\, dV$ as a triple integral in spherical coordinates.

You do not need to evaluate the integral!

A correct solution is: We first describe the region $\mathcal{E}$ in spherical coordinates. The cone and the plane are both rotationally symmetric around the z-axis, so these do not restrict the values of θ. Only the condition $y \leq 0$ restricts θ, and $y \leq 0$ corresponds to $\pi \leq \theta \leq 2\pi$.

The cone $z = \sqrt{3x^2 + 3y^2}$ can be described by $\rho\cos(\phi) = \sqrt{3\rho^2\sin(\phi)^2} = \sqrt{3}\rho\sin(\phi)$, since ρ and $\sin(\phi)$ are both positive. This equation yields $\cos(\phi) = \sqrt{3}\sin(\phi)$, i.e. $\tan(\phi) = \frac{1}{\sqrt{3}}$. This equation has only one solution with $0 \leq \phi \leq \pi$, which means that the cone can be described as $\phi = \frac{\pi}{6}$.

The plane $z = 2$ becomes $\rho\cos(\phi) = 2$, or $\rho = \frac{2}{\cos(\phi)}$.

The variable ρ starts at the origin, where $\rho = 0$ and ends at the plane. ϕ starts at the point $(0, 0, 2)$ where $\phi = 0$ and ends at the cone. Combining these yields that $\mathcal{E}$ can be described as

$$\mathcal{E} = \{(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi) \,|\, 0 \leq \rho \leq \frac{2}{\cos(\phi)}, 0 \leq \phi \leq \frac{\pi}{6}, \pi \leq \theta \leq 2\pi\}.$$

The integrand $\sqrt{x^2 + y^2 + z^2}$ becomes ρ in spherical coordinates. So by multiplying with the Jacobian $\rho^2\sin(\phi)$, we find

$$\iiint_{\mathcal{E}} \sqrt{x^2 + y^2 + z^2}\, dV = \int_{\pi}^{2\pi} \int_{0}^{\frac{\pi}{6}} \int_{0}^{\frac{2}{\cos(\phi)}} \rho^3 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$

Open questions

The next questions need to be worked out completely, every answer needs to be reasoned. Make the exercises in the box. If necessary, there is extra space at the back of the exam. If you use this extra space, clearly indicate the numbering of the questions there AND write in the regular answer box that you use the extra space. The maximum points per question is indicated in the margin.

7. (6 pt) Let $\mathcal{C}$ be the triangle with vertices $(0, 0)$, $(-1, 2)$ and $(-1, -2)$ with clockwise orientation. Evaluate $\oint_{\mathcal{C}} (2y - \cos(x))\, dx + \left(4x + e^{2y}\right)\, dy$.

A correct solution is: Let $\mathcal{D}$ be the region enclosed by $\mathcal{C}$. Then $\mathcal{C}$ has a negative orientation with respect to $\mathcal{D}$. Hence, we apply Green's theorem to the curve $-\mathcal{C}$. We obtain

$$\oint_{\mathcal{C}} (2y - \cos(x))\, dx + \left(4x + e^{2y}\right)\, dy = -\oint_{-\mathcal{C}} (2y - \cos(x))\, dx + \left(4x + e^{2y}\right)\, dy$$

$$= -\iint_{\mathcal{D}} \left(\frac{\partial(4x + e^{2y})}{\partial x} - \frac{\partial(2y - \cos(x))}{\partial y}\right) dA = -\iint_{\mathcal{D}} 2\, dA = -2\text{Area}(\mathcal{D}).$$

Since $\mathcal{D}$ is the filled triangle with vertices $(0, 0)$, $(-1, 2)$ and $(-1, -2)$ it has area 2. We conclude

$$\oint_{\mathcal{C}} (2y - \cos(x))\, dx + \left(4x + e^{2y}\right)\, dy = -2\text{Area}(\mathcal{D}) = -4.$$

8. (8 pt) Let $\mathcal{D}$ be the region in $\mathbb{R}^2$ bounded in between the parabola $y = x^2$ and the line $y = 2$. Find the absolute minimum and absolute maximum of the function $f(x, y) = 6x^4 + y^3 - 6y^2 + 9y$ on $\mathcal{D}$ and the points at which these values occur.

A correct solution is: We first find the critical points of f. For this we compute the partial derivatives

$$f_x = 24x^3, f_y = 3y^2 - 12y + 9.$$

Setting $f_x = 0$ yields $x = 0$, while setting $f_y = 0$ yields $y = 1$ or $y = 3$. So the critical points are $(0, 1)$ and $(0, 3)$. However, $(0, 3)$ is outside of the region $\mathcal{D}$, so we should discard it. For future reference, we compute $f(0, 1) = 4$.

The parabola $y = x^2$ and $y = 2$ intersect at the points $(-\sqrt{2}, 2)$ and $(\sqrt{2}, 2)$. We first consider the behaviour of f on $y = x^2$ for $-\sqrt{2} \leq x \leq \sqrt{2}$. We compute

$$f(x, x^2) = 6x^4 + x^6 - 6x^4 + 9x^2 = x^6 + 9x^2.$$

Differentiating this expression with respect to x yields $5x^5 + 18x$, which is only 0 for $x = 0$. The relevant points on this curve are, therefore, $(0, 0)$ as well as the end points $(-\sqrt{2}, 2)$ and $(\sqrt{2}, 2)$. We compute $f(0, 0) = 0$, $f(-\sqrt{2}, 2) = 26$ and $f(\sqrt{2}, 2) = 26$.

For the behaviour of f on $y = 2$ for $-\sqrt{2} \leq x \leq \sqrt{2}$, we compute

$$f(x, 2) = 6x^4 + 2.$$

Differentiating this expression with respect to x yields $24x^3$, which is only 0 for $x = 0$. The only new relevant point is, therefore, the point $(0, 2)$, where $f(0, 2) = 2$.

Since we have found all candidate locations for the absolute minimum and the absolute maximum, we only need to compare the computed function values. We find that f has an absolute minimum of 0 at $(0, 0)$ and it has an absolute maximum of 26 at $(-\sqrt{2}, 2)$ and $(\sqrt{2}, 2)$.

9. (6 pt) Consider the coordinate transformation $\begin{cases} u &= 2x + y \\ v &= x - 2y \end{cases}$. Let $\mathcal{D}$ be the region enclosed by the lines $y = 3 - 2x, y = 4 - 2x, y = \frac{x}{2} - 1$ and $y = \frac{x}{2} - 2$. Express and evaluate the integral $\iint_{\mathcal{D}} (3x - y)^2 \, dA$ using uv-coordinates. If needed, you may use that $\dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{1}{\frac{\partial(u, v)}{\partial(x, y)}}$.

A correct solution is: We first rewrite the lines $y = 3 - 2x, y = 4 - 2x, y = \frac{x}{2} - 1$ and $y = \frac{x}{2} - 2$ into forms that can more easily be rewritten into u, v-coordinates by writing them as $2x + y = 3, 2x + y = 4, x - 2y = 2$ and $x - 2y = 4$ respectively. In u, v-coordinates, these lines become $u = 3, u = 4, v = 2$ and $v = 4$ respectively.

The integrand is given in u, v-coordinates by $(3x - y)^2 = (u + v)^2$.

Finally, we compute the Jacobian $\dfrac{\partial(x, y)}{\partial(u, v)}$, which we can do in two different ways. We can either express x and y in terms of u and v by writing $x = \frac{2}{5}u + \frac{1}{5}v$ and $y = \frac{1}{5}u - \frac{2}{5}v$ and then computing $\dfrac{\partial(x, y)}{\partial(u, v)} = \frac{2}{5} \cdot (-\frac{2}{5}) - \frac{1}{5} \cdot \frac{1}{5} = -\frac{1}{5}$. Or, we can use the hint and write $\dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{1}{\frac{\partial(u, v)}{\partial(x, y)}} = \frac{1}{2 \cdot (-2) - 1 \cdot 1} = -\frac{1}{5}$.

Hence, we can compute

$$\iint_{\mathcal{D}} (3x - y)^2 \, dA = \int_3^4 \int_2^4 (u + v)^2 \left| -\frac{1}{5} \right| \, dv \, du = \frac{256}{15}.$$

10. (6pt) Let $\mathcal{C}$ be the curve in $\mathbb{R}^3$ that starts at the point $(1, 0, 0)$ and spirals once around the cylinder $x^2 + z^2 = 1$ along a circular helix in counterclockwise direction when viewed from the negative y-axis, and ends at the point $(1, 2\pi, 0)$. Consider the vector field $\mathbf{F} = \langle 2xyz, x^2z + 1 + x, x^2y \rangle$. Evaluate $\displaystyle\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

A correct solution is: The vector field $\mathbf{F}$ is not conservative, but it can be written as $\mathbf{F}_1 + \mathbf{F}_2$, where $\mathbf{F}_1 = \langle 2xyz, x^2z + 1, x^2y \rangle$ and $\mathbf{F}_2 = \langle 0, x, 0 \rangle$. Then $\mathbf{F}_1$ is conservative with potential function $f(x, y, z) = x^2yz + y$. Hence, with the fundamental theorem for line integrals we can evaluate

$$\oint_{\mathcal{C}} \mathbf{F}_1 \cdot d\mathbf{r} = f(1, 2\pi, 0) - f(1, 0, 0) = 2\pi - 0 = 2\pi.$$

For the line integral $\displaystyle\oint_{\mathcal{C}} \mathbf{F}_2 \cdot d\mathbf{r}$, we parametrize $\mathcal{C}$ as $\langle \cos(t), t, \sin(t) \rangle$ for $0 \leq t \leq 2\pi$. Then we can compute

$$\oint_{\mathcal{C}} \mathbf{F}_2 \cdot d\mathbf{r} = \int_0^{2\pi} \langle 0, \cos(t), 0 \rangle \cdot \langle -\sin(t), 1, \cos(t) \rangle \, dt = \int_0^{2\pi} \cos(t) \, dt = 0.$$

So in total, we obtain

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \oint_{\mathcal{C}} \mathbf{F}_1 \cdot d\mathbf{r} + \oint_{\mathcal{C}} \mathbf{F}_2 \cdot d\mathbf{r} = 2\pi.$$

Alternatively, we can also evaluate $\displaystyle\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ directly using the parametrization used above, by computing

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \big\langle 2\cos(t) t \sin(t), \cos^2(t)\sin(t) + 1 + \cos(t), \cos^2(t)\sin(t) \big\rangle \cdot \langle -\sin(t), 1, \cos(t) \rangle \, dt$$

$$= \ldots = 2\pi.$$

$$\textbf{Grade} = \frac{\text{obtained points}}{6} + 1$$