Exam 2 - EE1M1 Calculus (28/01/2025 09:00 - 11:00)

Fill in your personal information and
answer the seven questions in Grasple and
write down all your steps for the four open question and
submit in when finished.

Student number: ______________________

You are allowed to use:

- Pen, pencils and scrap paper.

Formula sheet

Some trigonometric formulae

$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$
$\cos(2\alpha) = 2\cos^2(\alpha) - 1 = 1 - 2\sin^2(\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$

Some limits

$$\lim_{x\to\infty} \frac{x^p}{e^x} = 0$$
$$\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x = e^a$$
$$\lim_{x\to\infty} \frac{\ln x}{x^p} = 0 \quad (p > 0)$$

Some integrals

$$\int \frac{\mathrm{d}\,x}{\sin(x)} = \ln\left|\tan\left(\frac{x}{2}\right)\right| + C$$
$$\int \frac{\mathrm{d}\,x}{\cos(x)} = \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$$
$$\int \frac{\mathrm{d}\,x}{1 + x^2} = \arctan(x) + C$$
$$\int \frac{\mathrm{d}\,x}{1 - x^2} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C$$
$$\int \frac{\mathrm{d}\,x}{\sqrt{1 - x^2}} = \arcsin(x) + C = -\arccos(x) + C$$
$$\int \frac{\mathrm{d}\,x}{\sqrt{x^2 + 1}} = \ln\bigl(x + \sqrt{x^2 + 1}\bigr) + C$$
$$\int \frac{\mathrm{d}\,x}{\sqrt{x^2 - 1}} = \ln\bigl|x + \sqrt{x^2 - 1}\bigr| + C$$
$$\int \sqrt{1 + x^2}\ \mathrm{d}\,x = \frac{1}{2}x\sqrt{1 + x^2} + \frac{1}{2}\ln\bigl(x + \sqrt{1 + x^2}\bigr) + C$$
$$\int \sqrt{1 - x^2}\ \mathrm{d}\,x = \frac{1}{2}x\sqrt{1 - x^2} + \frac{1}{2}\arcsin(x) + C$$
$$\int_0^{\frac{\pi}{2}} \sin^n x\ \mathrm{d}\,x = \begin{cases} \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\cdots\frac{3}{4}\frac{1}{2}\frac{\pi}{2} & \text{if } n \text{ even and } n \geq 2 \\ \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\cdots\frac{4}{5}\frac{2}{3} & \text{if } n \text{ odd and } n \geq 3 \end{cases}$$

Short-Answer questions (total: 44 points)

1. (1 + 4 pt) Consider the vector fields
$\mathbf{F}(x,y,z) = \langle 5x+2z, ze^{2yz}, 2x+ye^{2yz}\rangle$ and
$\mathbf{G}(x,y,z) = \langle xe^{2xz}+2y, 2x+4y, ze^{2xz}\rangle$.
(a.) Which of the two has a potential?
(b.) Give a potential of the vector field above which has a potential. (You do not have to add a constant of integration.)

(a.) ☐**F** ☐**G**

(b.)

2. (4 pt) Consider the function $g(x,y,z)$ of which the gradient vector at the point $(-4,4,-5)$ is given by

$$\nabla g(-4,4,-5) = \begin{bmatrix} -9 \\ 7 \\ -3 \end{bmatrix}.$$

Calculate the maximum value of the directional derivative of $g(x,y,z)$ at the point $(-4,4,-5)$.

3. (8 pt) Consider the function $f(x,y) = 3x^2y + x^2 + 8y^2$. The points $P = (0,0)$ and $Q = \left(\frac{4}{3}, -\frac{1}{3}\right)$. are critical points of f.
Use the second derivatives test to determine the type of the critical points P and Q.

P :
☐Local maximum ☐Local minimum
☐Saddle point ☐Second derivatives test is inconclusive

Q :
☐Local maximum ☐Local minimum
☐Saddle point ☐Second derivatives test is inconclusive

4. (4 + 4 pt) We want to evaluate $\int_{\mathcal{C}} (2xy^2+5)\,ds$ with $\mathcal{C}$ the curve on the circle $x^2+y^2=0$ from $(3,0)$ to $(0,3)$ in counterclockwise direction.
Our goal is to rewrite the line integral as a regular single integral. You do not need to evaluate the integral.
(a.) Give a parametrization $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ of $\mathcal{C}$ with $lb \le t \le ub$.
(b.) We write the integral as
$$\int_{\mathcal{C}} (2xy^2+5)\,ds = \int_{lb}^{ub} f(t)\,dt.$$
Give the integrand $f(t)$.

(a.)

$x(t) =$

$y(t) =$

$lb =$

$ub =$

(b.)

$f(t) =$

5. (9 pt) Let E be a solid. Suppose that the integral $\iiint_E f(x,y,z)\,dV$ can be written as the iterated integral $\int_0^2 \int_0^{5-\frac{5}{2}x} \int_0^{3-\frac{3}{5}z} f(x,y,z)\,dy\,dz\,dx$. Change the order of integration to write this integral as $\int_{\dots}^{\dots} \int_{\dots}^{\dots} \int_{\dots}^{\dots} f(x,y,z)\,dx\,dz\,dy$.

$$\int_0^2 \int_0^{5-\frac{5}{2}x} \int_0^{3-\frac{3}{5}z} f(x,y,z)\,dy\,dz\,dx =$$

6. (6 pt) Consider the surface defined by the equation $3x^2 + xy^3 - 2z^2 = 28$, and the point $P = (3,1,1)$ on this surface.
Find an equation for the tangent plane to S at P.

7. (4 pt) Given are the vector field $\mathbf{F}(x,y) = \langle x^2, 4x - 5y\rangle$ and the curve $\mathcal{C}$ with parametrization $\mathbf{r}(t) = \langle e^t, t^2\rangle$ from $(1,0)$ to $(e^3, 9)$.
Our goal is to evaluate the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ by writing it as an integral $\int_{lb}^{ub} f(t)\,dt$. You do not need to evaluate the integral.
Give the integrand $f(t)$, the lower limit lb and the upper limit ub.

$f(t) =$

$lb =$

$ub =$

Open questions (total: 40 points)
The next questions need to be worked out completely, every answer needs to be motivated. Write the solution in the box. If necessary, there is extra space at the end of the exam. If you use this extra space, clearly indicate the numbering of the questions there AND write in the regular answer box that you use the extra space. The maximum points per question is indicated in the margin.

8. (8 pt) Evaluate the double integral $\iint_{\mathcal{D}} (6x^2+4)(y+x^3)\, dA$, where $\mathcal{D}$ is the region bounded in between the lines $y+x^3=1$, $y+x^3=4$, $y-2x=1$ and $y-2x=4$.

9. (12 pt) Let $\mathcal{D}$ be the region in $\mathbb{R}^2$ inside the triangle with vertices $(-2,-1)$, $(2,3)$ and $(2,-1)$. Find the absolute minimum and absolute maximum of the function $f(x,y) = x^3 - 12xy + 6y^2$ on $\mathcal{D}$ and the points at which these values occur.

10. (6+6 pt) Let $\mathcal{E}$ be the solid region in $\mathbb{R}^3$ that lies below the cone $z = -\sqrt{3x^2 + 3y^2}$ and inside the sphere $x^2 + y^2 + (z+1)^2 = 1$, and has $x \geq 0$.

Write the triple integral $\iiint_{\mathcal{E}} x\sqrt{x^2 + y^2 + z^2}\, dV$ as an iterated integral in

(a.) cylindrical coordinates **and**

(b.) spherical coordinates.

In both cases, you do not need to evaluate the integral, i.e. you can leave your answers in the form $\int_{\dots}^{\dots} \int_{\dots}^{\dots} \int_{\dots}^{\dots} \dots\dots\; d\dots d\dots d\dots.$

(a.)

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(b.)

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11. (8 pt) Consider the vector field $\mathbf{F}(x, y) = \langle \sin(x^2)y + 2xy, x^2 \rangle$. Let $\mathcal{C}$ be the triangular curve consisting of the line segment from $(0, 0)$ to $(1, 2)$, followed by the line segment from $(1, 2)$ to $(1, 0)$, and the line segment from $(1, 0)$ back to $(0, 0)$.

Evaluate the line integral $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

$$\textbf{Grade} = \frac{\text{obtained points}}{84} \cdot 9 + 1$$

THE END

Extra space 1 (Clearly indicate which question this extra space relates to).

Extra space 2 (Clearly indicate which question this extra space relates to).