Exam 2 - EE1M1 Calculus (28/01/2025 09:00 - 11:00)

Fill in your personal information and
answer the seven questions in Grasple and
write down all your steps for the four open question and
submit in when finished.

Student number: ______________________

You are allowed to use:

- Pen, pencils and scrap paper.

How to start your exam:

1. Go to the **Brightspace** page of this course.
2. From 15 minutes before the scheduled start time:
   Click on the **link to Grasple** in the new **exam announcement.**
3. Log in using your **TU Delft credentials** (a.k.a. NetID).
   You should be able to do this without a password manager!
4. Open the provided test and click the **Launch Schoolyear browser** button.
5. Schoolyear will start, after which you again log in using your **TU Delft credentials**.
6. *From 5 minutes past the scheduled start time:*
   Start, perform and **submit the test** shown.

When you are finished with the exam, please follow these steps:

1. **Submit the exam** in Grasple.
2. **Close** Schoolyear.
3. Gather all your items and move **quietly** to the exit of the exam room.
4. Hand in **all scrap paper** at the exit.
5. Hand in this **exam sheet**.
6. Leave the exam room.

Formula sheet

Some trigonometric formulae

$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$
$\cos(2\alpha) = 2\cos^2(\alpha) - 1 = 1 - 2\sin^2(\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$

Some limits

$$\lim_{x\to\infty} \frac{x^p}{e^x} = 0$$
$$\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x = e^a$$
$$\lim_{x\to\infty} \frac{\ln x}{x^p} = 0 \quad (p > 0)$$

Some integrals

$$\int \frac{\mathrm{d}\,x}{\sin(x)} = \ln\left|\tan\left(\frac{x}{2}\right)\right| + C$$
$$\int \frac{\mathrm{d}\,x}{\cos(x)} = \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$$
$$\int \frac{\mathrm{d}\,x}{1 + x^2} = \arctan(x) + C$$
$$\int \frac{\mathrm{d}\,x}{1 - x^2} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C$$
$$\int \frac{\mathrm{d}\,x}{\sqrt{1 - x^2}} = \arcsin(x) + C = -\arccos(x) + C$$
$$\int \frac{\mathrm{d}\,x}{\sqrt{x^2 + 1}} = \ln\big(x + \sqrt{x^2 + 1}\big) + C$$
$$\int \frac{\mathrm{d}\,x}{\sqrt{x^2 - 1}} = \ln\big|x + \sqrt{x^2 - 1}\big| + C$$
$$\int \sqrt{1 + x^2}\ \mathrm{d}\,x = \frac{1}{2}x\sqrt{1 + x^2} + \frac{1}{2}\ln\big(x + \sqrt{1 + x^2}\big) + C$$
$$\int \sqrt{1 - x^2}\ \mathrm{d}\,x = \frac{1}{2}x\sqrt{1 - x^2} + \frac{1}{2}\arcsin(x) + C$$
$$\int_0^{\frac{\pi}{2}} \sin^n x\ \mathrm{d}\,x = \begin{cases} \dfrac{n-1}{n}\dfrac{n-3}{n-2}\dfrac{n-5}{n-4}\cdots\dfrac{3}{4}\dfrac{1}{2}\dfrac{\pi}{2} & \text{if } n \text{ even and } n \geq 2 \\ \dfrac{n-1}{n}\dfrac{n-3}{n-2}\dfrac{n-5}{n-4}\cdots\dfrac{4}{5}\dfrac{2}{3} & \text{if } n \text{ odd and } n \geq 3 \end{cases}$$

Short-Answer questions (total: 44 points)
Note: The short-answer questions on Grasple are parametrized, meaning that your version of the exercise may contain different numbers than those presented in the solutions below. In any case, the ideas behind the calculations remain the same.

1. (1 + 4 pt) Consider the vector fields
$\mathbf{F}(x,y,z) = \langle 5x + 2z, ze^{2yz}, 2x + ye^{2yz}\rangle$ and
$\mathbf{G}(x,y,z) = \langle xe^{2xz} + 2y, 2x + 4y, ze^{2xz}\rangle$.
(a.) Which of the two has a potential?
(b.) Give a potential of the vector field above which has a potential. (You do not have to add a constant of integration.)

A correct solution is:
(a.) Suppose a vector field $\mathbf{H} = \langle P, Q, R\rangle$ has a potential ϕ, i.e. $\phi_x = P$, $\phi_y = Q$ $\phi_z = R$. Clairaut's theorem now tells us that

$$P_y = \phi_{xy} = \phi_{yx} = Q_x.$$

Similarly, we find that $P_z = R_x$ and $Q_z = R_y$. In particular, the functions $Q_x - P_y$, $P_z - R_x$ and $R_y - Q_z$ should all be 0 when $\mathbf{H}$ has a potential.

Here $\frac{\partial}{\partial z}\left(xe^{2xz} + 2y\right) - \frac{\partial}{\partial x}\left(ze^{2x}\right) = 2(x^2 - z^2)e^{2xz} \neq 0$, so $\mathbf{G}$ cannot be conservative. On the other hand, for $\mathbf{F}$ all three the corresponding functions are 0 and in the upcoming subquestion we will construct a potential for $\mathbf{F}$.
(b.) To find a potential ϕ we have to solve $\dfrac{\partial \phi}{\partial x}(x,y,z) = 5x + 2z$, $\dfrac{\partial \phi}{\partial y}(x,y,z) = ze^{2yz}$, and $\dfrac{\partial \phi}{\partial z}(x,y,z) = 2x + ye^{2yz}$. From the first equation we find

$$\phi(x,y,z) = \int (5x + 2z)\, dx = \frac{25}{2}x^2 + 2xz + C(y,z)$$

where the constant of integration $C(y,z)$ is allowed to depend on y and z.

Plugging this ϕ into the second equation we get

$$ze^{2yz} = \frac{\partial \phi}{\partial y}(x,y,z) = \frac{\partial C}{\partial y}(y,z)$$

which means $\dfrac{\partial C}{\partial y}(y,z) = ze^{2yz}$, so $C(y,z) = \frac{e^{2yz}}{2} + D(z)$ where $D(z)$ does not depend on y.

Plugging $\phi(x,y,z) = 2x + ye^{2yz} + \dfrac{dD}{dz}(z)$ into the final equation we get

$$2x + ye^{2yz} = \frac{\partial \phi}{\partial z}(x,y,z) = 2x + ye^{2yz} + \frac{dD}{dz}(z)$$

This means $0 = \frac{dD}{dz}(z)$, so $D(z) = E$, where E is a constant which does not depend on either x, y or z.

Therefore, a potential is given by $\phi(x,y,z) = \frac{5x^2}{2} + 2xz + \frac{e^{2yz}}{2}$. All other potentials differ from this one by a constant.

2. (4 pt) Consider the function $g(x, y, z)$ of which the gradient vector at the point $(-4, 4, -5)$ is given by

$$\nabla g(-4, 4, -5) = \begin{bmatrix} -9 \\ 7 \\ -3 \end{bmatrix}.$$

Calculate the maximum value of the directional derivative of $g(x, y, z)$ at the point $(-4, 4, -5)$.

A correct solution is: The maximum value of the directional derivative is $|\nabla f(-4, 4, -5)|$. Thus

$$|\nabla f(-4, 4, -5)| = \sqrt{(-9)^2 + (7)^2 + (-3)^2} = \sqrt{139}$$

3. (8 pt) Consider the function $f(x, y) = 3x^2y + x^2 + 8y^2$. The points $P = (0, 0)$ and $Q = \left(\frac{4}{3}, -\frac{1}{3}\right)$. are critical points of f.
Use the second derivatives test to determine the type of the critical points P and Q.

A correct solution is: In order to apply the second derivatives test, we first compute the partial derivatives of f. We obtain $f_x(x, y) = 6xy + 2x$ and $f_y(x, y) = 3x^2 + 16y$. Next we compute the second derivatives of f to obtain $f_{xx}(x, y) = 6y + 2$, $f_{xy}(x, y) = f_{yx}(x, y) = 6x$ and $f_{yy}(x, y) = 16$. To classify these critical points, we need to evaluate the function $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - f_{x,y}(x, y)^2$ at these points.

For the point P we obtain $D(0, 0) = 2 \cdot 16 - 0^2 = 32$. Since $D(0, 0) > 0$ and $f_{xx}(0, 0) = 2 > 0$, the second derivatives test tells us that f attains a local minimum at P.

For the point Q we obtain $D\left(\frac{4}{3}, -\frac{1}{3}\right) = 0 \cdot 16 - 8^2 = -64$. Since $D\left(\frac{4}{3}, -\frac{1}{3}\right) < 0$ the second derivatives test us that Q is a saddle point.

4. (4 + 4 pt) We want to evaluate $\int_{\mathcal{C}} (2xy^2 + 5)\, ds$ with $\mathcal{C}$ the curve on the circle $x^2 + y^2 = 0$ from $(3, 0)$ to $(0, 3)$ in counterclockwise direction.
Our goal is to rewrite the line integral as a regular single integral. You do not need to evaluate the integral.
(a.) Give a parametrization $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ of $\mathcal{C}$ with $lb \leq t \leq ub$.
(b.) We write the integral as $\int_{\mathcal{C}} (2xy^2 + 5)\, ds = \int_{lb}^{ub} f(t)\, dt$.
Give the integrand $f(t)$.

A correct solution is:
(a.) A convenient way to parametrize (a part of) a circle with radius r in counterclockwise direction is to use $x(t) = r\cos(t)$ and $y(t) = r\sin(t)$. Therefore, we can choose $x(t) = 3\cos(t)$ and $y(t) = 3\sin(t)$. In order to ensure that the curve starts at $(3, 0)$ and ends at $(0, 3)$, we should choose $lb = 0$ and $ub = \frac{\pi}{2}$.

(b.) In general, a line integral can be evaluated as

$$\int_{\mathcal{C}} g(x, y)\, ds = \int_{lb}^{ub} g(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}\, dt$$

Substituting $x = 3\cos(t)$ and $y = 3\sin(t)$ into $2xy^2 + 5$, we obtain the expression

$54\sin^2(t)\cos(t)+5$. In addition, we can compute

$$\sqrt{x'(t)^2+y'(t)^2}=\sqrt{(-3\sin(t))^2+(3\cos(t))^2}=3.$$

So, we must have $f(t)=3(54\sin^2(t)\cos(t)+5)$.

5. (9 pt) Let E be a solid. Suppose that the integral $\iiint_E f(x,y,z)\,dV$ can be written as the iterated integral $\int_0^2\int_0^{5-\frac{5}{2}x}\int_0^{3-\frac{3}{5}z} f(x,y,z)\,dy\,dz\,dx$. Change the order of integration to write this integral as $\int_{\dots}^{\dots}\int_{\dots}^{\dots}\int_{\dots}^{\dots} f(x,y,z)\,dx\,dz\,dy$.

A correct solution is: Inspecting the original iterated integral, the limits of the innermost integral (over y) yield that the region E is bounded by the planes $y=0$ and $y=3-\frac{3}{5}z$. From the limits of the middle integral (over z) we can deduce that the lowest value of z is $z=0$. We find that in the new iterated integral the lower limit for y is 0, while the upper limit is $3-\frac{3}{5}\cdot 0=3$.

Inspecting the original iterated integral, the limits of the innermost integral (over y) yield that the region E is bounded by the planes $y=0$ and $y=3-\frac{3}{5}z$, i.e. $z=5-\frac{5y}{3}$. From the limits of the middle integral (over z) we can deduce that the region is also bounded by the plane $z=0$ (the value $z=5-\frac{5}{2}x$ will, in the new integration order, be taken care of in the innermost integral over x). We find that in the new iterated integral the lower limit for z is 0, while the upper limit is $5-\frac{5y}{3}$.

Inspecting the original iterated integral, the limits of the middle integral (over z) yield that the region E is bounded by the planes $z=0$ and $z=5-\frac{5}{2}x$, i.e. $x=2-\frac{2z}{5}$. From the limits of the outer integral (over x) we can deduce that the region is also bounded by the plane $x=0$ (the value $x=2$ is consistent with the intersection point of the planes $z=0$ and $z=5-\frac{5}{2}x$). The limits of the inner integral (over y) do not involve x. We find that in the new iterated integral the lower limit for x is 0, while the upper limit is $2-\frac{2z}{5}$.

In total we find $\int_0^2\int_0^{5-\frac{5}{2}x}\int_0^{3-\frac{3}{5}z} f(x,y,z)\,dy\,dz\,dx=\int_0^3\int_0^{5-\frac{5y}{2}}\int_0^{2-\frac{2z}{5}} f(x,y,z)\,dx\,dz\,dy.$

6. (6 pt) Consider the surface defined by the equation $3x^2+xy^3-2z^2=28$, and the point $P=(3,1,1)$ on this surface.
Find an equation for the tangent plane to S at P.

A correct solution is: Taking $F(x,y,z)=3x^2+xy^3-2z^2$, we can think of the surface as a level surface of F. Then $\nabla F(P)$ is a normal vector to the tangent plane.

We have $\nabla F(x,y,z)=\begin{bmatrix}6x+y^3\\3xy^2\\-4z\end{bmatrix}$ and $\nabla F(3,1,1)=\begin{bmatrix}19\\9\\-4\end{bmatrix}$. The equation of the tangent plane to the surface can be calculated as $\nabla F(P)\cdot\begin{bmatrix}x-3\\y-1\\z-1\end{bmatrix}=0$, which leads to the equation

$19x + 9y - 4z = 62.$

It is possible to multiply both sides of this equation by a constant factor. This does not affect the tangent plane it describes.

7. (4 pt) Given are the vector field $\mathbf{F}(x, y) = \langle x^2, 4x - 5y\rangle$ and the curve $\mathcal{C}$ with parametrization $\mathbf{r}(t) = \langle e^t, t^2\rangle$ from $(1, 0)$ to $(e^3, 9)$.

Our goal is to evaluate the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ by writing it as an integral $\int_{lb}^{ub} f(t)\, dt$.
You do not need to evaluate the integral.
Give the integrand $f(t)$, the lower limit lb and the upper limit ub.

A correct solution is: In general, a line integral can be evaluated as

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{lb}^{ub} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt.$$

Substituting $x = e^t$ and $y = t^2$ into $\langle x^2, 4x - 5y\rangle$, we obtain the expression $\langle e^{2t}, -5t^2 + 4e^t\rangle$. In addition, we can compute $\mathbf{r}'(t) = \langle e^t, 2t\rangle$. So, we must have $f(t) = \langle e^{2t}, -5t^2 + 4e^t\rangle \cdot \langle e^t, 2t\rangle = -10t^2 + 8te^t + e^{3t}$. Finally, the point $(1, 0)$ corresponds to $t = 0$, so $lb = 0$, while the point $(e^3, 9)$ corresponds to $t = 3$, so $ub = 3$.

Open questions (total: 40 points)
The next questions need to be worked out completely, every answer needs to be motivated. Write the solution in the box. If necessary, there is extra space at the end of the exam. If you use this extra space, clearly indicate the numbering of the questions there AND write in the regular answer box that you use the extra space. The maximum points per question is indicated in the margin.

8. (8 pt) Evaluate the double integral $\iint_{\mathcal{D}} (6x^2+4)(y+x^3)\,dA$, where $\mathcal{D}$ is the region bounded in between the lines $y+x^3=1$, $y+x^3=4$, $y-2x=1$ and $y-2x=4$.

A correct solution is: We evaluate this double integral using the coordinate transformation

$$\begin{cases} u &= y+x^3 \\ v &= y-2x \end{cases}$$

The region $\mathcal{D}$ transforms into the rectangle $\mathcal{S}=\{(u,v)\,|\,1\le u\le 4, 1\le y\le 4\}$. We can compute the Jacobian as

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}} = \frac{1}{3x^2\cdot 1-1\cdot(-2)} = \frac{1}{3x^2+2}$$

In particular, we obtain

$$\iint_{\mathcal{D}} (6x^2+4)(y+x^3)\,dA = \iint_{\mathcal{S}} (6x^2+4)(y+x^3)\frac{1}{3x^2+2}\,du\,dv$$

$$= \int_1^4\int_1^4 2(y+x^3)\,du\,dv = \int_1^4\int_0^4 2u\,du\,dv$$

$$= \int_1^4 15\,dv = 45.$$

9. (12 pt) Let $\mathcal{D}$ be the region in $\mathbb{R}^2$ inside the triangle with vertices $(-2,-1)$, $(2,3)$ and $(2,-1)$. Find the absolute minimum and absolute maximum of the function $f(x,y)=x^3-12xy+6y^2$ on $\mathcal{D}$ and the points at which these values occur.

A correct solution is: We aim to find all candidate points where the minimum and maximum might occur. By comparing the function values at these points, we obtain where the minimum and maximum actually occur.

First, we find all critical points. For this we evaluate

$$f_x(x,y)=3x^2-12y, \qquad f_y(x,y)=-12x+12y$$

Setting $f_y(x,y)=0$ gives $x=y$. Plugging this into $f_x(x,y)=0$ gives $3x^2=12x$, so $x=0$ or $x=4$. As such, the critical points are $(0,0)$ and $(4,4)$. Of these, only $(0,0)$ lies inside the region, so we discard $(4,4)$.

Now we analyse the boundaries of the region. First we consider the line $y=-1, -2\le x\le 2$. Plugging $y=-1$ into f, gives the function $f(x,-1)=x^3+12x+6$. Computing

$\frac{d}{dx}f(x,-1) = 3x^2 + 12 = 0$ yields no further solutions. So the only new candidates are the corner points $(-2,-1)$ and $(2,-1)$.

Next, we consider the line $x = 2, -1 \leq y \leq 3$. Plugging $x = 2$ into f yields $f(2,y) = 8 - 24y + 6y^2$. Computing $\frac{d}{dy}f(2,y) = -24 + 12y$ gives $y = 2$. So the new candidate points are $(2,2)$ and the corner point $(2,3)$.

Finally, we consider the line $y = x + 1$ for $-2 \leq x \leq 2$. Plugging $y = x + 1$ into f yields $f(x, x+1) = x^3 - 12x(x+1) + 6(x+1)^2 = x^3 - 6x^2 + 6$. Computing $\frac{d}{dx}x^3 - 6x^2 + 6 = 3x^2 - 12x = 0$ gives $x = 0$ or $x = 4$. The point $(4,5)$ is outside the triangle, so we discard it. Hence, the only new candidate point is $(0,1)$.

Now we have found all candidates. We evaluate f at each point to obtain:

$$f(0,0) = 0, f(-2,-1) = -26, f(2,-1) = 38, f(2,2) = -16, f(2,3) = -10, f(0,1) = 6$$

We can conclude that the absolute minimum of f is -26 at the point $(-2,-1)$ and the absolute maximum of f is 38 at $(2,-1)$.

10. (6 + 6 pt) Let $\mathcal{E}$ be the solid region in $\mathbb{R}^3$ that lies below the cone $z = -\sqrt{3x^2 + 3y^2}$ and inside the sphere $x^2 + y^2 + (z+1)^2 = 1$, and has $x \geq 0$.
Write the triple integral $\iiint_{\mathcal{E}} x\sqrt{x^2 + y^2 + z^2}\, dV$ as an iterated integral in

(a.) cylindrical coordinates **and**

(b.) spherical coordinates.

In both cases, you do not need to evaluate the integral, i.e. you can leave your answers in the form $\int_{\dots}^{\dots}\int_{\dots}^{\dots}\int_{\dots}^{\dots} \dots\dots\, d\dots d\dots d\dots.$

A correct solution is:
(a.) In cylindrical coordinates, it can be seen that, as seen from below, the variable z starts at the sphere and ends at the cone. The cone can be written in cylindrical coordinates as

$$z = -\sqrt{3x^2 + 3y^2} = -\sqrt{3r^2} = -r\sqrt{3}$$

For the sphere, we first rewrite it to

$$(z+1)^2 = 1 - x^2 - y^2 = 1 - r^2$$

Taking the negative square root (since we are looking at the lower half of the sphere), we find

$$z + 1 = -\sqrt{1 - r^2} \qquad \Rightarrow \qquad z = -1 - \sqrt{1 - r^2}$$

In particular, we find $-1 - \sqrt{1 - r^2} \leq z \leq -r\sqrt{3}$.
The lowest value of r occurs at the z-axis, where $r = 0$. The highest value of r occurs when the sphere and the cone intersect. This happens when

$$1 = x^2 + y^2 + (z+1)^2 = x^2 + y^2 + (-\sqrt{3x^2 + 3y^2} + 1)^2 = r^2 + (-\sqrt{3}r + 1)^2 = 4r^2 - 2\sqrt{3}r + 1,$$

i.e. when $r = \frac{1}{2}\sqrt{3}$ (since $r > 0$).

Finally, the condition $x \geq 0$ means that $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. There are no further restrictions on θ. In total, not forgetting the Jacobian r, we find

$$\iiint_{\mathcal{E}} x\sqrt{x^2+y^2+z^2}\,dV = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\frac{1}{2}\sqrt{3}}\int_{-1-\sqrt{1-r^2}}^{-r\sqrt{3}} r\cos(\theta)\sqrt{r^2+z^2}\,r\,dz\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\frac{1}{2}\sqrt{3}}\int_{-1-\sqrt{1-r^2}}^{-r\sqrt{3}} r^2\cos(\theta)\sqrt{r^2+z^2}\,dz\,dr\,d\theta$$

(b.) In spherical coordinates the condition $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ holds as well.

The cone can be written as

$$\rho\cos(\phi) = -\sqrt{3(\rho\sin(\phi)\cos(\theta)^2) + 3(\rho\sin(\phi)\sin(\theta))^2} = -\sqrt{3\rho^2\sin^2(\phi)(\cos^2(\theta)+\sin^2(\theta)}$$

$$= -\sqrt{3\rho^2\sin^2(\phi)} = -\rho\sin(\phi)\sqrt{3}$$

where we use that $\sin(\phi) \geq 0$, since $0 \leq \phi \leq \pi$. In particular, we obtain

$$\cos(\phi) = -\sin(\phi)\sqrt{3},$$

which, for $0 \leq \phi \leq \pi$ only holds for $\phi = \frac{5\pi}{6}$. So ϕ starts at the cone, and ends at the negative z-axis, so we find $\frac{5\pi}{6} \leq \phi \leq \pi$.

The sphere can be written out as $x^2 + y^2 + z^2 + 2z = 0$, which in spherical coordinates becomes

$$\rho^2 + 2\rho\cos(\phi) = 0$$

So, this becomes $\rho = -2\cos(\phi)$. In particular, ρ starts at the origin (notice that the top of the sphere is at the origin) and ends at the sphere, so we have $0 \leq \rho \leq -2\cos(\phi)$.
In total, not forgetting the Jacobian $\rho^2\sin(\phi)$, we find

$$\iiint_{\mathcal{E}} x\sqrt{x^2+y^2+z^2}\,dV = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\frac{5\pi}{6}}^{\pi}\int_0^{-2\cos(\phi)} \rho\sin(\phi)\cos(\theta)\rho\cdot\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\frac{5\pi}{6}}^{\pi}\int_0^{-2\cos(\phi)} \rho^4\sin^2(\phi)\cos(\theta)\,d\rho\,d\phi\,d\theta$$

11. (8 pt) Consider the vector field $\mathbf{F}(x,y) = \langle \sin(x^2)y + 2xy, x^2\rangle$. Let $\mathcal{C}$ be the triangular curve consisting of the line segment from $(0,0)$ to $(1,2)$, followed by the line segment from $(1,2)$ to $(1,0)$, and the line segment from $(1,0)$ back to $(0,0)$.
Evaluate the line integral $\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$.

A correct solution is: We apply Green's theorem. The region $\mathcal{D}$ inside $\mathcal{C}$ can be described as

$$\mathcal{D} = \{(x,y)\,|\,0 \leq x \leq 1, 0 \leq y \leq 2x\}$$

Notice that $\mathcal{C}$ has a clockwise, i.e. negative, orientation with respect to $\mathcal{D}$. Writing $P(x,y) = \sin(x^2)y + 2xy$ and $Q(x,y) = x^2$, Green's theorem gives

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = -\iint_{\mathcal{D}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = -\int_0^1 \int_0^{2x} (2x - (\sin(x^2) + 2x))\, dy\, dx$$

$$= \int_0^1 \int_0^{2x} \sin(x^2)\, dy\, dx = \int_0^1 2x \sin(x^2)\, dx$$

$$= \int_0^1 \sin(u)\, du = [-\cos(u)]_0^1 = -\cos(1) + 1$$

where, in the final line, we used the substitution $u = x^2$.

$$\textbf{Grade} = \frac{\text{obtained points}}{84} \cdot 9 + 1$$

THE END