# Practice Exam with Grasple part - EE1M1 Calculus

**You are allowed to use:**

- Pen, pencils and scrap paper.

**The formula sheet can be found on the next page.**

# Formula sheet

### Some trigonometric formulae

$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$
$\cos(2\alpha) = 2\cos^2(\alpha) - 1 = 1 - 2\sin^2(\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$

### Some limits

$$\lim_{x\to\infty} \frac{x^p}{e^x} = 0$$
$$\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x = e^a$$
$$\lim_{x\to\infty} \frac{\ln x}{x^p} = 0 \quad (p > 0)$$

### Some integrals

$$\int \frac{\mathrm{d}\,x}{\sin(x)} = \ln\left|\tan\left(\frac{x}{2}\right)\right| + C$$
$$\int \frac{\mathrm{d}\,x}{\cos(x)} = \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$$
$$\int \frac{\mathrm{d}\,x}{1+x^2} = \arctan(x) + C$$
$$\int \frac{\mathrm{d}\,x}{1-x^2} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C$$
$$\int \frac{\mathrm{d}\,x}{\sqrt{1-x^2}} = \arcsin(x) + C = -\arccos(x) + C$$
$$\int \frac{\mathrm{d}\,x}{\sqrt{x^2+1}} = \ln\big(x + \sqrt{x^2+1}\big) + C$$
$$\int \frac{\mathrm{d}\,x}{\sqrt{x^2-1}} = \ln\big|x + \sqrt{x^2-1}\big| + C$$
$$\int \sqrt{1+x^2}\;\mathrm{d}\,x = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\ln\big(x + \sqrt{1+x^2}\big) + C$$
$$\int \sqrt{1-x^2}\;\mathrm{d}\,x = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin(x) + C$$
$$\int_0^{\frac{\pi}{2}} \sin^n x\;\mathrm{d}\,x = \begin{cases} \dfrac{n-1}{n}\dfrac{n-3}{n-2}\dfrac{n-5}{n-4}\cdots\dfrac{3}{4}\dfrac{1}{2}\dfrac{\pi}{2} & \text{if } n \text{ even and } n \geq 2 \\ \dfrac{n-1}{n}\dfrac{n-3}{n-2}\dfrac{n-5}{n-4}\cdots\dfrac{4}{5}\dfrac{2}{3} & \text{if } n \text{ odd and } n \geq 3 \end{cases}$$

**Grasple questions**
The first seven questions should be made on Grasple, using the following link.

Make sure to follow the input format for each exercise. Although only the final answer is graded, partial credits can be awarded for certain partially correct answers.

**Open questions**
The next questions need to be worked out completely, every answer needs to be reasoned.

8. Consider a differentiable function $f(x, y, z)$. Suppose the directional derivative of $f$ at the point $(3, 1, 4)$ in the direction of $\mathbf{u} = \langle 1, 2, 2\rangle$ is equal to 3 and suppose that the directional derivative of $f$ at $(3, 1, 4)$ is minimal in the direction of $\mathbf{v} = \langle -2, 0, -1\rangle$. Find $\nabla f(3, 1, 4)$.

   **A correct solution is:** Since the directional derivative is minimal in the direction of $-\nabla f(3, 1, 4)$, we must have that $\nabla f(3, 1, 4) = c\,\langle 2, 0, 1\rangle$ for some constant $c > 0$. We use the given information on $D_{\mathbf{u}} f(3, 1, 4)$ to find $c$. Indeed, we note that $\hat{\mathbf{u}} = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right\rangle$ is the unit vector in the direction of $\mathbf{u}$. As such, we must have that

$$3 = D_{\hat{\mathbf{u}}} f(3, 1, 4) = \hat{\mathbf{u}} \cdot \nabla f(3, 1, 4) = \frac{1}{3} \cdot (2c) + \frac{2}{3} \cdot 0 + \frac{2}{3} \cdot c = \frac{4}{3}c$$

   Hence, we find that $c = \frac{9}{4}$ and $\nabla f(3, 1, 4) = \frac{9}{4}\langle 2, 0, 1\rangle = \left\langle \frac{9}{2}, 0, \frac{9}{4}\right\rangle$.

9. Let $\mathcal{E}$ be the solid region inside the sphere $x^2 + y^2 + (z - 1)^2 = 1$ that has $y \leq 0$. Assume that a charge is distributed over $\mathcal{E}$ with charge density function $\sigma(x, y, z) = (x^2 + y^2 + z^2)^{\frac{3}{2}}$ Find the net total charge on $\mathcal{E}$.

   **A correct solution is:** The region can conveniently be described in spherical coordinates. The equation for the sphere can be rewritten as $x^2 + y^2 + z^2 - 2z = 0$, which means that in spherical coordinates it is given by $\rho^2 - 2\rho\cos(\phi) = 0$, i.e. $\rho = 2\cos(\phi)$. The condition $y \leq 0$ translates to $\pi \leq \theta \leq 2\pi$. Since $\mathcal{E}$ a sphere centered around $(0, 0, 1)$ with radius 1, it lies above the $xy$-plane and touches the $xy$-plane at the origin, so we have $0 \leq \phi \leq \frac{\pi}{2}$. As such, we obtain that the net total charge $q$ is given by

$$q = \iiint_{\mathcal{E}} q(x, y, z)\, dV = \int_{\pi}^{2\pi} \int_{0}^{\frac{\pi}{2}} \int_{0}^{2\cos(\phi)} \rho^2 \cdot \rho^3 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= \int_{\pi}^{2\pi} \int_{0}^{\frac{\pi}{2}} \frac{1}{6}(2\cos(\phi))^6 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= \int_{\pi}^{2\pi} \int_{1}^{0} -\frac{32}{3}u^6\, d\rho\, d\phi\, d\theta$$

$$\int_{\pi}^{2\pi} \frac{32}{21}\, d\theta = \frac{32\pi}{21}$$

   In this computation we used the substitution $u = \cos(\phi)$ with $du = -\sin(\phi)\, d\phi$.

10. Evaluate the iterated integral

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{0}\int_{x^2+y^2}^{2x^2+2y^2}(z-x^2-y^2)^{\frac{3}{2}}\,dz\,dy\,dx$$

**A correct solution is:** We recognize this iterated integral as a triple integral in cylindrical coordinates. Indeed, the condition $x^2+y^2 \le z \le 2x^2+2y^2$ translates to $r^2 \le z \le 2r^2$, while the region $-2 \le x \le 2, -\sqrt{4-x^2} \le y \le 0$ is the lower half of the filled disc with radius 2 centered around the origin, i.e. it has $\pi \le \theta \le 2\pi$ and $0 \le r \le 2$. As such, we find

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{0}\int_{x^2+y^2}^{2x^2+2y^2}(z-x^2-y^2)^{\frac{3}{2}}\,dz\,dy\,dx = \int_{\pi}^{2\pi}\int_{0}^{2}\int_{r^2}^{2r^2}(z-r^2)^{\frac{3}{2}}r\,dz\,dr\,d\theta$$

$$= \int_{\pi}^{2\pi}\int_{0}^{2}\left(\frac{2}{5}(2r^2-r^2)^{\frac{5}{2}}-\frac{2}{5}(r^2-r^2)^{\frac{5}{2}}\right)r\,dr\,d\theta = \int_{\pi}^{2\pi}\int_{0}^{2}\frac{2}{5}r^6\,dr\,d\theta$$

$$= \int_{\pi}^{2\pi}\frac{256}{35}\,d\theta = \frac{256\pi}{35}$$

11. Let $\mathbf{F}(x,y)$ be a conservative vector field on $\mathbb{R}^2$. In addition, consider the vector field $\mathbf{G}(x,y) = \langle xy, -xy\rangle$. Let $\mathcal{D}$ be the region bounded in between the lines $x+y=1$, $x+y=2$, $x-y=3$ and $x-y=5$ and let $\partial\mathcal{D}$ denote the boundary of $\mathcal{D}$ with clockwise orientation.
Evaluate $\int_{\partial\mathcal{D}}(\mathbf{F}(x,y)+\mathbf{G}(x,y))\cdot d\mathbf{r}$.

**A correct solution is:** We first note that since $\mathbf{F}$ is conservative, we can conclude from the fundamental theorem for line integrals that $\int_{\partial\mathcal{D}}\mathbf{F}(x,y)\cdot d\mathbf{r} = 0$. For the line integral of the vector field $\mathbf{G}$ we use Green's theorem. Note that $\partial\mathcal{D}$ has a negative orientation with respect to $\mathcal{D}$, so we find

$$\int_{\partial\mathcal{D}}\mathbf{G}(x,y)\cdot d\mathbf{r} = -\iint_{\mathcal{D}}\left(\frac{\partial(-xy)}{\partial x}-\frac{\partial(xy)}{\partial y}\right)dA = \iint_{\mathcal{D}}(y+x)\,dA$$

In order to evaluate the double integral, we apply to coordinate transformation $u = x+y, v = x-y$, since using these coordinates the region can conveniently be described as $1 \le u \le 2$, $3 \le v \le 5$. The Jacobian for this transformation is given by

$$\frac{\partial(u,v)}{\partial(x,y)} = 1\cdot(-1) - 1\cdot 1 = -2$$

As such, we find

$$\iint_{\mathcal{D}}(y+x)\,dA = \int_1^2\int_3^5 u\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dv\,du$$

$$= \int_1^2\int_3^5 u\frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|}\,dv\,du = \int_1^2\int_3^5 u\frac{1}{2}\,dv\,du$$

$$= \int_1^2 u\,du = \frac{3}{2}$$

We conclude that $\int_{\partial\mathcal{D}}(\mathbf{F}(x,y)+\mathbf{G}(x,y))\cdot d\mathbf{r} = \frac{3}{2}$.

THE END