TU Delft

Practice Exam 2 - EE1M1 Calculus

You are allowed to use:

- Pen, pencils and scrap paper.

The formula sheet can be found on the next page.

Formula sheet

Some trigonometric formulae

$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$
$\cos(2\alpha) = 2\cos^2(\alpha) - 1 = 1 - 2\sin^2(\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$

Some limits

$$\lim_{x\to\infty} \frac{x^p}{e^x} = 0$$
$$\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x = e^a$$
$$\lim_{x\to\infty} \frac{\ln x}{x^p} = 0 \quad (p > 0)$$

Some integrals

$$\int \frac{\mathrm{d}\,x}{\sin(x)} = \ln\left|\tan\left(\frac{x}{2}\right)\right| + C$$
$$\int \frac{\mathrm{d}\,x}{\cos(x)} = \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$$
$$\int \frac{\mathrm{d}\,x}{1+x^2} = \arctan(x) + C$$
$$\int \frac{\mathrm{d}\,x}{1-x^2} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C$$
$$\int \frac{\mathrm{d}\,x}{\sqrt{1-x^2}} = \arcsin(x) + C = -\arccos(x) + C$$
$$\int \frac{\mathrm{d}\,x}{\sqrt{x^2+1}} = \ln\bigl(x + \sqrt{x^2+1}\bigr) + C$$
$$\int \frac{\mathrm{d}\,x}{\sqrt{x^2-1}} = \ln\bigl|x + \sqrt{x^2-1}\bigr| + C$$
$$\int \sqrt{1+x^2}\ \mathrm{d}\,x = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\ln\bigl(x + \sqrt{1+x^2}\bigr) + C$$
$$\int \sqrt{1-x^2}\ \mathrm{d}\,x = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin(x) + C$$
$$\int_0^{\frac{\pi}{2}} \sin^n x\ \mathrm{d}\,x = \begin{cases} \dfrac{n-1}{n}\dfrac{n-3}{n-2}\dfrac{n-5}{n-4}\cdots\dfrac{3}{4}\dfrac{1}{2}\dfrac{\pi}{2} & \text{if } n \text{ even and } n \geq 2 \\ \dfrac{n-1}{n}\dfrac{n-3}{n-2}\dfrac{n-5}{n-4}\cdots\dfrac{4}{5}\dfrac{2}{3} & \text{if } n \text{ odd and } n \geq 3 \end{cases}$$

Short-answer questions

An explanation is not required for the short-answer questions. Only the answer matters. You do not need to fully simplify your answers.

1. Is the vector field $\mathbf{F}(x,y,z) = \langle -2y\sin(2xy), -2x\sin(2xy), \cos(2xy)\rangle$ conservative? If it is conservative, give a potential function.

2. Consider the function $f(x,y) = e^{x-y}$. For which direction $\mathbf{u}$ does $D_{\mathbf{u}}f(2,2)$ reach its minimal value?

3. Let $\mathcal{C}$ be the curve that first follows the straight line from $(1,3)$ to $(-2,4)$ and then the parabola $y = x^2$ to $(-3,9)$ and consider the vector field $\mathbf{F} = \langle 2xy - y^2, x^2 - 2xy\rangle$. Evaluate the line integral $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$.

4. Reverse the order of integration for $\int_{-4}^{0}\int_{3}^{\sqrt{25-x^2}} f(x,y)\,dy\,dx$ and give the resulting integral.

5. Consider the function $f(x,y) = x^2y + x^2 + 2y^2 - 3$. Give the coordinates of all critical points of f that correspond to local minima, local maxima and saddle points (if they exist).

   Local Minima:

   Local maxima:

   Saddle points:

6. A charge density $q(x,y,z)$ is distributed over the region E in between the cone $z = 3\sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2 = 4$. Express the net total charge as a triple integral in cylindrical coordinates.

7. Consider a lamina on a bounded region $\mathcal{D}$ with constant density K and total mass m and let $\overline{x}$ denote the x-coordinate of the center of mass of the lamina. Let $\mathcal{C}$ denote the boundary curve of $\mathcal{D}$. Find a vector field $\mathbf{F}$ for which $\overline{x} = \displaystyle\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

Open questions

The next questions need to be worked out completely, every answer needs to be reasoned.

8. Let D be the triangle with vertices $(-1, 0)$, $(1, 1)$ and $(1, -1)$. Find the absolute minimum and absolute maximum of the function $f(xy) = 2x^2 - 3xy$ on D.

9. Consider the coordinate transformation $\begin{cases} u &= \sqrt{x-y} \\ v &= \sqrt{x+y} \end{cases}$. Let $\mathcal{D}$ be the region enclosed by the lines $y = 1-x, y = 4-x, y = x-1$ and $y = x-4$. Express and evaluate the integral $\displaystyle\iint_{\mathcal{D}} e^{x+y}\, dA$ using uv-coordinates. If needed, you may use that $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}}$.

10. Consider the function $f(x, y)$ with the property that $f(-x, y) = -f(x, y)$. We are given that $\iint_{\mathcal{D}} (3 + 2f(x, y))\, dA = 8$. Furthermore, $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$, where $\mathcal{D}_1$ and $\mathcal{D}_2$ are non-overlapping regions each with surface area equal to 2. Also, the region $\mathcal{D}_1$ is symmetric with respect to reflection in the y-axis. Evaluate the integral $\iint_{\mathcal{D}_2} f(x, y)\, dA$.

11. Let $\mathcal{E}$ be the region in between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$. Evaluate $\iiint_{\mathcal{E}} (x^2+y^2)\, dV$.

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