TU Delft

Practice Exam 2 - EE1M1 Calculus

You are allowed to use:

- Pen, pencils and scrap paper.

The formula sheet can be found on the next page.

Formula sheet

Some trigonometric formulae

$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$
$\cos(2\alpha) = 2\cos^2(\alpha) - 1 = 1 - 2\sin^2(\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$

Some limits

$$\lim_{x\to\infty} \frac{x^p}{e^x} = 0$$
$$\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x = e^a$$
$$\lim_{x\to\infty} \frac{\ln x}{x^p} = 0 \quad (p > 0)$$

Some integrals

$$\int \frac{\mathrm{d}\,x}{\sin(x)} = \ln\left|\tan\left(\frac{x}{2}\right)\right| + C$$
$$\int \frac{\mathrm{d}\,x}{\cos(x)} = \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$$
$$\int \frac{\mathrm{d}\,x}{1+x^2} = \arctan(x) + C$$
$$\int \frac{\mathrm{d}\,x}{1-x^2} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C$$
$$\int \frac{\mathrm{d}\,x}{\sqrt{1-x^2}} = \arcsin(x) + C = -\arccos(x) + C$$
$$\int \frac{\mathrm{d}\,x}{\sqrt{x^2+1}} = \ln\big(x + \sqrt{x^2+1}\big) + C$$
$$\int \frac{\mathrm{d}\,x}{\sqrt{x^2-1}} = \ln\big|x + \sqrt{x^2-1}\big| + C$$
$$\int \sqrt{1+x^2}\ \mathrm{d}\,x = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\ln\big(x + \sqrt{1+x^2}\big) + C$$
$$\int \sqrt{1-x^2}\ \mathrm{d}\,x = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin(x) + C$$
$$\int_0^{\frac{\pi}{2}} \sin^n x\ \mathrm{d}\,x = \begin{cases} \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\cdots\frac{3}{4}\frac{1}{2}\frac{\pi}{2} & \text{if } n \text{ even and } n \geq 2 \\ \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\cdots\frac{4}{5}\frac{2}{3} & \text{if } n \text{ odd and } n \geq 3 \end{cases}$$

Short-answer questions

An explanation is not required for the short-answer questions. Only the answer matters. You do not need to fully simplify your answers.

1. Is the vector field $\mathbf{F}(x, y, z) = \langle -2y\sin(2xy), -2x\sin(2xy), \cos(2xy)\rangle$ conservative? If it is conservative, give a potential function.

   A correct solution is: If ϕ were a potential for $\mathbf{F}$, then, since $\dfrac{\partial \phi}{\partial x} = -2y\sin(2xy)$, we should have $\phi = \cos(2xy) + h(y, z)$, where h is a function of only y and z. However, since $\dfrac{\partial \phi}{\partial z} = \cos(2xy)$, we should have $\dfrac{\partial h}{\partial z} = \cos(2xy)$, which is impossible since h does not depend on x. So $\mathbf{F}$ is not conservative.

2. Consider the function $f(x, y) = e^{x-y}$. For which direction $\mathbf{u}$ does $D_{\mathbf{u}}f(2, 2)$ reach its minimal value?

   A correct solution is: The directional derivative takes its minimal value in the direction of $-\nabla f(2, 2)$. Since $\nabla f = \langle e^{x-y}, -e^{x-y}\rangle$, we find $-\nabla f(2, 2) = \langle -1, 1\rangle$. Since $\mathbf{u}$ should be a unit vector, we find that $\mathbf{u} = \dfrac{-\nabla f(2, 2)}{\|\nabla f(2, 2)\|} = \langle -\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\rangle$.

3. Let $\mathcal{C}$ be the curve that first follows the straight line from $(1, 3)$ to $(-2, 4)$ and then the parabola $y = x^2$ to $(-3, 9)$ and consider the vector field $\mathbf{F} = \langle 2xy - y^2, x^2 - 2xy\rangle$. Evaluate the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

   A correct solution is: The vector field $\mathbf{F}$ is conservative, since $\phi = x^2y - xy^2$ is a potential function. Hence, the fundamental theorem for line integrals allows us to directly evaluate the line integral by evaluating the potential at the start and end point of the curve (the integral can also be evaluated by parametrizing the curve, but that is more work). This gives

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \phi(-3, 9) - \phi(1, 3) = 330.$$

4. Reverse the order of integration for $\displaystyle\int_{-4}^{0}\int_{3}^{\sqrt{25-x^2}} f(x, y)\, dy\, dx$ and give the resulting integral.

   A correct solution is: The region of integration is $\{(x, y) \mid -4 \le x \le 0, 3 \le y \le \sqrt{25 - x^2}\}$, i.e. the part of the circle centered around the origin with radius 5 with $y \ge 3$ and $x \le 0$. Reversing the order of integration, we find $\displaystyle\int_{-4}^{0}\int_{3}^{\sqrt{25-x^2}} f(x, y)\, dy\, dx = \int_{3}^{5}\int_{-\sqrt{25-y^2}}^{0} f(x, y)\, dx\, dy$.

5. Consider the function $f(x, y) = x^2y + x^2 + 2y^2 - 3$. Give the coordinates of all critical points of f that correspond to local minima, local maxima and saddle points (if they exist).

   A correct solution is: The partial derivatives of f are given by $f_x = 2x(y + 1)$ and $f_y = x^2 + 4y$. Since the partial derivatives are defined everywhere, we only need to consider points where both partial derivatives are 0. Solving $f_x = 0$ yields $x = 0$ or $y = -1$. The system $x = 0, f_y = 0$ gives $y = 0$, while the system $y = -1, f_y = 0$ gives $x = 2$ or $x = -2$. So f has three critical points $(0, 0)$, $(2, -1)$ and $(-2, -1)$. The second order partial derivatives are given by $f_{xx} = 2(y + 1)$, $f_{xy} = f_{yx} = 2x$ and $f_{yy} = 4$. So the discriminant D satisfies $D(0, 0) = 8$, $D(2, -1) = -16$ and $D(-2, -1) = 16$. Since

$f_{xx}(0,0) > 0$, we find that f has a local minimum at $(0,0)$, while $(2,-1)$ and $(-2,-1)$ are saddle points.

6. A charge density $q(x,y,z)$ is distributed over the region E in between the cone $z = 3\sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2 = 4$. Express the net total charge as a triple integral in cylindrical coordinates.
   A correct solution is: In cylindrical coordinates, the cone is given by $z = 3r$, while the sphere is given by $r^2+z^2 = 4$. Since the sphere is above the cone, we can take the positive square root to find $z = \sqrt{4-r^2}$. The sphere and the cone intersect when $3r = \sqrt{4-r^2}$, i.e. when $r = \sqrt{\frac{2}{5}}$. Finally, the region is rotationally symmetric around the z-axis. We find that the total charge Q satisfies

$$Q = \int_0^{2\pi} \int_0^{\sqrt{\frac{2}{5}}} \int_{3r}^{\sqrt{4-r^2}} q(r\cos(\theta), r\sin(\theta), z) r \, dz \, dr \, d\theta.$$

7. Consider a lamina on a bounded region $\mathcal{D}$ with constant density K and total mass m and let $\overline{x}$ denote the x-coordinate of the center of mass of the lamina. Let $\mathcal{C}$ denote the boundary curve of $\mathcal{D}$. Find a vector field $\mathbf{F}$ for which $\overline{x} = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.
   A correct solution is: The x-coordinate of the center of mass is given by the double integral

$$\overline{x} = \iint_{\mathcal{D}} K \frac{x}{m} \, dA.$$

   According to Green's theorem, we have for any vector field $\mathbf{F} = \langle P, Q \rangle$ that

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

   So we need to find a vector field for which $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = K \frac{x}{m}$. There are many vector fields with this property, including $\mathbf{F} = \left\langle 0, K\frac{x^2}{2m} \right\rangle$ and $\mathbf{F} = \left\langle -K\frac{xy}{m}, 0 \right\rangle$ and any of these will suffice.

Open questions

The next questions need to be worked out completely, every answer needs to be reasoned.

8. Let D be the triangle with vertices $(-1,0)$, $(1,1)$ and $(1,-1)$. Find the absolute minimum and absolute maximum of the function $f(x,y) = 2x^2 - 3xy$ on D.
   A correct solution is: The partial derivative of f are given by $f_x = 4x - 3y$ and $f_y = -3x$. Solving $f_y = 0$ yields $y = 0$ and plugging this into the equation $f_x = 0$ gives that $(0,0)$ is the only critical point of f. Moreover, we have $f(0,0) = 0$.

   In order to find the other candidate locations for the absolute maximum and minimum, we evaluate f at the boundaries of the region. First we consider the line piece from $(-1,0)$ to $(1,1)$, i.e. $y = \frac{x}{2} + \frac{1}{2}$. We compute $f\left(x, \frac{x}{2} + \frac{1}{2}\right) = \frac{x^2}{2} - \frac{3x}{2}$. This function has a local minimum at $x = \frac{3}{2}$, which is outside the region under consideration. Now we consider the line piece from $(-1,0)$ to $(1,-1)$, i.e. $y = -\frac{x}{2} - \frac{1}{2}$. We compute $f\left(x, -\frac{x}{2} - \frac{1}{2}\right) = \frac{7x^2}{2} + \frac{3x}{2}$. This function has a local minimum at $x = -\frac{3}{14}$, which corresponds to $y = -\frac{17}{28}$, and we have $f\left(\frac{-3}{14}, -\frac{17}{28}\right) = -\frac{9}{56}$. Finally, we consider the line piece from $(1,-1)$ to $(1,1)$, i.e. $x = 1$. We compute $f(1,y) = 2 - 3y$, which does not have any local minima or maxima.

To conclude we consider the edge points of the region and compute the function values there: $f(-1,0)=2$, $f(1,1)=-1$ and $f(1,-1)=5$.

Since these are all the candidates for the locations of the absolute minimum and the absolute maximum, the largest function value corresponds to the absolute maximum and the smallest function value corresponds to the absolute minimum. We find that the absolute minimum is -1 at $(1,1)$ and the absolute maximum is 5 at $(1,-1)$.

9. Consider the coordinate transformation $\begin{cases} u &= \sqrt{x-y} \\ v &= \sqrt{x+y} \end{cases}$. Let $\mathcal{D}$ be the region enclosed by the lines $y=1-x, y=4-x, y=x-1$ and $y=x-4$. Express and evaluate the integral $\iint_{\mathcal{D}} e^{x+y}\, dA$ using uv-coordinates. If needed, you may use that $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$.

   A correct solution is: We note that $u^2 = x-y$ and $v^2 = x+y$, so that $x = \frac{1}{2}\left(u^2+v^2\right)$ and $y = \frac{1}{2}\left(v^2-u^2\right)$. Using these, we can compute

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} u & v \\ -u & v \end{vmatrix} = 2uv.$$

   In uv-coordinates, the limit $y=1-x$, $y=4-x$, $y=x-1$ and $y=x-4$ transform into $v^2=1, v^2=4, u^2=1$ and $u^2=4$ respectively. Since u and v are positive, we obtain the limits $1 \le u \le 2$ and $1 \le v \le 2$. We conclude that

$$\iint_{\mathcal{D}} e^{x+y}\, dA = \int_1^2 \int_1^2 2uve^{v^2}\, dv\, du = \frac{3}{2}\left(e^4 - e\right).$$

10. Consider the function $f(x,y)$ with the property that $f(-x,y) = -f(x,y)$. We are given that $\iint_{\mathcal{D}} (3+2f(x,y))\, dA = 8$. Furthermore, $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$, where $\mathcal{D}_1$ and $\mathcal{D}_2$ are non-overlapping regions each with surface area equal to 2. Also, the region $\mathcal{D}_1$ is symmetric with respect to reflection in the y-axis. Evaluate the integral $\iint_{\mathcal{D}_2} f(x,y)\, dA$.

    A correct solution is: Since the double integral of the constant function 1, is the area of integration, we find that

$$\iint_{\mathcal{D}} 3\, dA = 3\text{Area}(\mathcal{D}) = 12,$$

    so that $\iint_{\mathcal{D}} 2f(x,y)\, dA = 8-12 = -4$, which yields that $\iint_{\mathcal{D}} f(x,y)\, dA = -2$. Since $f(-x,y) = -f(x,y)$ and $\mathcal{D}_1$ is symmetric with respect to reflection in the y-axis, we find that the part integrating f over the part of $\mathcal{D}_1$ to the left of the y-axis cancels out the integral over the part of $\mathcal{D}_1$ to the right of the y-axis, so $\iint_{\mathcal{D}_1} f(x,y)\, dA = 0$. We conclude that $\iint_{\mathcal{D}_2} f(x,y)\, dA = -2$.

11. Let $\mathcal{E}$ be the region in between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$. Evaluate $\iiint_{\mathcal{E}} (x^2+y^2)\, dV$.

    A correct solution is: We evaluate the integral in spherical coordinates. The spheres turn into the equations $\rho = 1$ and $\rho = 2$ respectively. Since the region $\mathcal{E}$ is rotationally symmetric in all directions, the limits of the integral become $1 \leq \rho \leq 2$, $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$. Finally, the integrand $x^2 + y^2$ becomes $\rho^2 \sin^2(\phi)$. Upon multiplying by the Jacobian $\rho^2 \sin(\phi)$, we obtain

$$\begin{aligned}
\iiint_{\mathcal{E}} (x^2 + y^2)\, dV &= \int_0^{2\pi} \int_0^{\pi} \int_1^2 \rho^4 \sin^3(\phi)\, d\rho d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} \int_1^2 \rho^4 \sin(\phi)(1 - \cos^2(\phi))\, d\rho d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} \frac{31}{5} \sin(\phi)(1 - \cos^2(\phi))\, d\phi d\theta \\
&= \int_0^{2\pi} \int_1^{-1} -\frac{31}{5}(1 - u^2)\, du d\theta \\
&= \int_0^{2\pi} \frac{124}{15} d\theta \\
&= \frac{248}{15}\pi.
\end{aligned}$$

In this computation we used the substitution $u = \cos(\phi)$.

THE END