Exam 1 - EE1M1 Calculus (12/12/2023 09:00 - 11:00)

Fill in your personal information and
write down your answers for the eight short-answer questions and
write down all your steps for the five open question and
hand in when finished.

You are allowed to use:

- Pen, pencils and scrap paper;
- A simple calculator;
- The formula sheet;

Short-answer questions

An explanation is not required for the short-answer questions. Only the answer matters. The maximum points per question is indicated in the margin.

Clearly write the answer in the box. You do not need to fully simplify your answers.

1. (3 pt) Let $f(x, y) = 2x^2y - 2y^3$. Find an equation of the tangent plane to the graph of f at the point $(2, -1, -6)$.

   A correct solution is: The tangent plane is given by $z = f(2, -1) + f_x(2, -1)(x - 2) + f_y(2, -1)(y + 1)$. The partial derivatives of f can be computed as $f_x(x, y) = 4xy$ and $f_y(x, y) = 2x^2 - 6y^2$. In the point $(2, -1)$ they are given by $f_x(2, -1) = -8$ and $f_y(2, -1) = 2$. Therefore, the tangent plane is given by $z = -6 - 8(x - 2) + 2(y + 1)$.

2. (4 pt) Evaluate the limit

$$\lim_{x\to-\infty} \sqrt{4x^2 - 3x + 1} + 2x + 1.$$

   A correct solution is: We use the 'square root trick' to rewrite the limit into

$$\begin{aligned}\lim_{x\to-\infty} \sqrt{4x^2 - 3x + 1} + 2x + 1 &= \lim_{x\to-\infty} \left(\sqrt{4x^2 - 3x + 1} + 2x + 1\right) \tfrac{\sqrt{4x^2-3x+1}-2x-1}{\sqrt{4x^2-3x+1}-2x-1} \\ &= \lim_{x\to-\infty} \frac{4x^2 - 3x + 1 - (2x + 1)^2}{\sqrt{4x^2 - 3x + 1} - 2x - 1} \\ &= \lim_{x\to-\infty} \frac{-7x}{\sqrt{4x^2 - 3x + 1} - 2x - 1}.\end{aligned}$$

   Now we divide by the dominant term x, but in order to take this term inside the square root, we have to use that $\frac{1}{x} = -\frac{1}{-x} = -\frac{1}{\sqrt{x^2}}$ since x is negative. Hence, we obtain

$$\begin{aligned}\lim_{x\to-\infty} \sqrt{4x^2 - 3x + 1} + 2x + 1 &= \lim_{x\to-\infty} \frac{-7x}{\sqrt{4x^2 - 3x + 1} - 2x - 1} \\ &= \lim_{x\to-\infty} \frac{-7}{-\sqrt{4 - \frac{3}{x} + \frac{1}{x^2}} - 2 - \frac{1}{x}} \\ &= \frac{-7}{-\sqrt{4} - 2} = \frac{7}{4}.\end{aligned}$$

3. (4 pt) Simplify $\tan(\arccos(x))$ into an expression that does not involve trigonometric functions or their inverses.

   A correct solution is: To help us understand this problem, we first define $y = \arccos(x)$. Note that y is an angle. By definition of the arccosine, y is the unique angle such that $\cos(y) = x$, and $0 \leq y \leq \pi$. For this problem, we need to figure out $\tan(\arccos(x)))$, which is $\tan(y)$ for the angle defined above. We draw the point $(\cos(y), \sin(y))$ on the unit circle. This point can be drawn using the two properties of y stated above. In this figure, y corresponds to the counterclockwise angle from the positive x-axis to the line segment between the origin to the point $(\cos(y), \sin(y))$. The absolute value of the tangent is equal to the length of the vertical side divided by the length of the horizontal side. We know that the hypotenuse has length 1, and the horizontal side has length $\cos(y) = x$. Applying the Pythagorean Theorem, we obtain $\sin(y) = \text{length of vertical side} = \sqrt{1^2 - x^2}$. Therefore, the tangent has value $\tan(\arccos(x)) = \tan(y) = \frac{\sqrt{1-x^2}}{x}$.

4. (4 pt) Consider the curve defined by
$y - e^y = x^2 y$ in $\mathbb{R}^2$. Determine $\dfrac{dy}{dx}$.

A correct solution is: We differentiate this equation implicitly to obtain

$$\frac{dy}{dx} - e^y \frac{dy}{dx} = 2xy + x^2 \frac{dy}{dx}.$$

Here, we applied the chain rule when differentiating the exponential function and the product rule when differentiating $x^2 y$. Rewriting this equation to make sure all terms with $\frac{dy}{dx}$ are on the same side yields

$$\left(1 - e^y - x^2\right) \frac{dy}{dx} = 2xy$$

and upon dividing by the factor in front of $\frac{dy}{dx}$ we obtain

$$\frac{dy}{dx} = \frac{2xy}{1 - e^y - x^2}.$$

5. (4 pt) Let $z(s,t) = f(u(s,t), v(s,t))$, where f, u, and v are differentiable with the following values given:

Function	Value at$(1,4)$	Value at$(7,9)$
f	3	9
f_u	8	3
f_v	-2	1
u	7	4
u_s	6	7
u_t	-1	-3
v	9	1
v_s	4	9
v_t	8	4

So, for example, $u(1,4) = 7$.
Find $z_s(1,4)$.

A correct solution is: We apply the chain rule to find

$$z_s(1,4) = f_u(u(1,4), v(1,4))u_s(1,4) + f_v(u(1,4), v(1,4))v_s(1,4).$$

Plugging in the correct values we obtain

$$z_s(1,4) = f_u(7,9) \cdot 6 + f_v(7,9) \cdot 4 = 3 \cdot 6 + 1 \cdot 4 = 22.$$

6. (6 pt) Evaluate the integral

$$\int \sin(x)\cos(x)\ln\left(\cos(x)\right)\, dx.$$

A correct solution is: We apply the substitution $u = \cos(x)$. With $du = -\sin(x)dx$, this yields

$$\int \sin(x)\cos(x)\ln\left(\cos(x)\right)\, dx = \int -u\ln\left(u\right)\, du.$$

Now we apply integration by parts. By differentiating the logarithm and integrating $-u$, we obtain

$$\int -u \ln(u)\, du = -\frac{1}{2}u^2 \ln(u) - \int -\frac{1}{2}u^2 \frac{1}{u}\, du = -\frac{1}{2}u^2 \ln(u) + \frac{1}{4}u^2 + C.$$

Pluggin back in $u = \cos(x)$, we conclude

$$\int \sin(x)\cos(x)\ln(\cos(x))\, dx = -\frac{1}{2}\cos^2(x)\ln(\cos(x)) + \frac{1}{4}\cos^2(x) + C.$$

7. (4 pt) Find the angle (in radians) between the vectors $\mathbf{u} = \langle 2, 0, -2\rangle$ and $\mathbf{v} = \langle 2, 1, -1\rangle$.

A correct solution is: The angle θ satisfies

$$\cos(\theta) = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{6}{\sqrt{8}\sqrt{6}} = \frac{1}{2}\sqrt{3}.$$

Therefore, we must have $\theta = \frac{\pi}{6}$.

8. (4 pt) Evaluate the limit

$$\lim_{x\to 0^+} x^{2x/(\ln(x)\sin(x))}.$$

A correct solution is: The limit is an indeterminate form. We start by writing the power as an exponential function and taking the limit inside this exponential function to obtain

$$\lim_{x\to 0^+} x^{2x/(\ln(x)\sin(x))} = \lim_{x\to 0^+} \left(e^{\ln(x)}\right)^{2x/(\ln(x)\sin(x))} = \lim_{x\to 0^+} e^{2x/\sin(x)} = e^{\lim\limits_{x\to 0^+} \frac{2x}{\sin(x)}}.$$

Now, either using a standard limit or l'Hospital's rule, we find that $\lim\limits_{x\to 0^+} \frac{2x}{\sin(x)} = 2$. Therfore, we conclude that

$$\lim_{x\to 0^+} x^{2x/(\ln(x)\sin(x))} = e^{\lim_{x\to 0^+} \frac{2x}{\sin(x)}} = e^2.$$

Open questions

The next questions need to be worked out completely, every answer needs to be reasoned. Make the exercises in the box. If necessary, there is extra space at the back of the exam. If you use this extra space, clearly indicate the numbering of the questions there AND write in the regular answer box that you use the extra space. The maximum points per question is indicated in the margin.

9. (6 pt) Consider the com[illegible] polynomial $p(z) =$ [illegible] $4z^2 + 14z - 20$. One of the roots of $p(z)$ is $z = 2$. Find all roots [illegible] his polynomia[illegible] peated according to their multiplicity).

A correct solution is: S[illegible] z [illegible]s a root of p, the factorization of p contains a factor $(z-2)$. Performing a long di[illegible], we can factor $p(z) = (z-2)(z^2 - 2z + 10)$. Either by using the "ABC-formula" o[illegible] leting the square, we find that the quadratic equation $z^2 - 2z + 10 = 0$ has ro[illegible] $= 1 -$ [illegible] nd $z = 1 + 3i$. So all roots are $z = 2, z = 1 - 3i$ and $z = 1 + 3i$.

10. (6 pt) Let A be the point $(1, 0, 1)$ and let V be the plane through the points $(2, 1, 3)$, $(3, 1, 3)$ and $(2, 0, 2)$. Find the distance between A and V.

A correct solution is: The plane V is spanned by the vectors $\mathbf{u} = \langle 1, 0, 0\rangle$, the vector from $(2, 1, 3)$ to $(3, 1, 3)$, and $\mathbf{v} = \langle 0, -1, -1\rangle$, the vector from $(2, 1, 3)$ to $(2, 0, 2)$. As such, a normal vector for the plane is given by $\mathbf{n} = \mathbf{u} \times \mathbf{v} = \langle 0, 1, -1\rangle$. We write B for the point on V closest to A. Then the vector $\overrightarrow{AB}$ is perpendicular to the plane V. In particular, $\mathbf{AB}$ is a scalar multiple of the normal $\mathbf{n}$. Let $\mathbf{w} = \langle 1, 1, 2\rangle$ be the vector from A to the point $(2, 1, 3)$, which we will call P, on V. Then we can write $\mathbf{w} = \overrightarrow{AB} + \overrightarrow{BP}$. Since $\overrightarrow{AB}$ and $\overrightarrow{BP}$ are perpendicular, we must have that $\overrightarrow{AB}$ is the orthogonal projection of $\mathbf{w}$ onto the normal $\mathbf{n}$. As such, we can compute

$$\overrightarrow{AB} = \mathrm{proj}_{\mathbf{n}}(\mathbf{w}) = \frac{\mathbf{w} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}\mathbf{n} = \frac{-1}{2}\mathbf{n} = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Since B is the point on V closest to A, we find that the distance between A and V is given by

$$\|\overrightarrow{AB}\| = \frac{1}{2}\sqrt{2}.$$

11. (8 pt) Evaluate the integral $\displaystyle\int \frac{x^2 + 3x + 4}{(x - 1)(x^2 + 2x + 5)}\, dx.$

A correct solution is: We apply partial fraction decomposition. The degree of the numerator is less than the degree of the denominator, so we do not need to apply long division. The denominator is the product of a linear factor and an irreducible quadratic factor. Hence, we should write

$$\frac{x^2 + 3x + 4}{(x - 1)(x^2 + 2x + 5)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 2x + 5} = \frac{Ax^2 + 2Ax + 5A + Bx^2 - Bx + Cx - C}{(x - 1)(x^2 + 2x + 5)}$$

for unknown constants A, B, C. We obtain the linear system of equations by comparing the coefficients of each power of x

$$\begin{cases} 1 &= A + B \\ 3 &= 2A - B + C \\ 4 &= 5A - C \end{cases}$$

The first equation yields $B = 1 - A$, while the third equation yields $C = 5A - 4$. Plugging these into the second equation gives $3 = 2A - (1 - A) + (5A - 4) = 8A - 5$, from which it follows that $A = 1$. Hence, we obtain $B = 0$ and $C = 1$. Thus we can write

$$\frac{x^2 + 3x + 4}{(x - 1)(x^2 + 2x + 5)} = \frac{1}{x - 1} + \frac{1}{x^2 + 2x + 5}.$$

The first of these terms can be integrated as usual. For the second term, we complete the square and divide both parts of the fraction by 4 to write

$$\frac{1}{x^2 + 2x + 5} = \frac{1}{(x + 1)^2 + 4} = \frac{\frac{1}{4}}{\left(\frac{x+1}{2}\right)^2 + 1}.$$

Now, in order to integrate the second term, we use the substitution $u = \frac{x+1}{2}$ with $du = \frac{1}{2}dx$ to obtain

$$\begin{aligned}
\int \frac{x^2+3x+4}{(x-1)(x^2+2x+5)}\,dx &= \int \frac{1}{x-1}\,dx + \int \frac{\frac{1}{4}}{\left(\frac{x+1}{2}\right)^2+1}\,dx \\
&= \ln|x-1| + \int \frac{\frac{1}{2}}{u^2+1}\,du \\
&= \ln|x-1| + \tfrac{1}{2}\arctan(u) + C \\
&= \ln|x-1| + \tfrac{1}{2}\arctan\left(\tfrac{x+1}{2}\right) + C.
\end{aligned}$$

12. (6 pt) Determine whether the following integral is convergent or divergent, and in case of convergence compute the integral

$$\int_2^\infty \frac{x^2+3}{\sqrt{x^6-3x^2}}\,dx.$$

A correct solution is: We apply the comparison test. By dividing both parts of the fraction by x^2, we note that the function behaves as $\frac{1}{x}$ as x becomes large. In particular, we find

$$\frac{x^2+3}{\sqrt{x^6-3x^2}} \geq \frac{x^2}{\sqrt{x^6-3x^2}} \geq \frac{x^2}{\sqrt{x^6}} = \frac{1}{x}.$$

Since the integral $\int_2^\infty \frac{1}{x}\,dx$ diverges, the comparison test yields that $\int_2^\infty \frac{x^2+3}{\sqrt{x^6-3x^2}}$ diverges as well.

13. (4pt) Show that the equ[redacted] $x^3 - 8x + 6 = 0$ [redacted] at least one solution between 0 and 1.

A correct solution is[redacted]he fu[redacted]n $f(x) = x^3 - 8x + 6$ is continuous and satisfies $f(0) = 6$ and $f(1) = -1$.[redacted]ermediate Value Theorem yields that there exists a number c with $0 < c < 1$ an[redacted] $= 0$, since 0 is in between 6 and -1. In particular, for this number c we have c^3[redacted]$+$[redacted]0, as desired.

$$\textbf{Grade} = \frac{\text{obtained points}}{7} + 1$$