Practice Exam 1 - EE1M1 Calculus

You are allowed to use:

- Pen, pencils and scrap paper.

The formula sheet can be found on the next page.

Formula sheet

Some trigonometric formulae

$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$
$\cos(2\alpha) = 2\cos^2(\alpha) - 1 = 1 - 2\sin^2(\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$

Some limits

$$\lim_{x\to\infty} \frac{x^p}{e^x} = 0$$
$$\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x = e^a$$
$$\lim_{x\to\infty} \frac{\ln x}{x^p} = 0 \quad (p > 0)$$

Some integrals

$$\int \frac{\mathrm{d}\,x}{\sin(x)} = \ln\left|\tan\left(\frac{x}{2}\right)\right| + C$$
$$\int \frac{\mathrm{d}\,x}{\cos(x)} = \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$$
$$\int \frac{\mathrm{d}\,x}{1 + x^2} = \arctan(x) + C$$
$$\int \frac{\mathrm{d}\,x}{1 - x^2} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C$$
$$\int \frac{\mathrm{d}\,x}{\sqrt{1 - x^2}} = \arcsin(x) + C = -\arccos(x) + C$$
$$\int \frac{\mathrm{d}\,x}{\sqrt{x^2 + 1}} = \ln\left(x + \sqrt{x^2 + 1}\right) + C$$
$$\int \frac{\mathrm{d}\,x}{\sqrt{x^2 - 1}} = \ln\left|x + \sqrt{x^2 - 1}\right| + C$$
$$\int \sqrt{1 + x^2}\ \mathrm{d}\,x = \frac{1}{2}x\sqrt{1 + x^2} + \frac{1}{2}\ln\left(x + \sqrt{1 + x^2}\right) + C$$
$$\int \sqrt{1 - x^2}\ \mathrm{d}\,x = \frac{1}{2}x\sqrt{1 - x^2} + \frac{1}{2}\arcsin(x) + C$$
$$\int_0^{\frac{\pi}{2}} \sin^n x\ \mathrm{d}\,x = \begin{cases} \dfrac{n-1}{n}\dfrac{n-3}{n-2}\dfrac{n-5}{n-4}\cdots\dfrac{3}{4}\dfrac{1}{2}\dfrac{\pi}{2} & \text{if } n \text{ even and } n \geq 2 \\ \dfrac{n-1}{n}\dfrac{n-3}{n-2}\dfrac{n-5}{n-4}\cdots\dfrac{4}{5}\dfrac{2}{3} & \text{if } n \text{ odd and } n \geq 3 \end{cases}$$

Short-answer questions

An explanation is not required for the short-answer questions. Only the answer matters. You do not need to fully simplify your answers.

1. Find all horizontal and vertical asymptotes of the function $f(x) = \dfrac{(x-5)\sqrt{x^2+1}+2x-10}{x^2-2x-15}$.

2. Consider $f(x) = \sqrt{x}$. Give the sharpest upper bound for the Lagrange remainder when approximating $\sqrt{16.02}$ using a Taylor polynomial of degree 3 with center 16.

3. Evaluate the integral
$$\int \arcsin(2x)\, dx.$$

4. Find an equation of the tangent line to the curve
$$x^3 - \cos\left(\frac{2\pi x}{y}\right) = yx^2 + 3$$
at the point $(2, 1)$.

5. Recall that the equation of an ellipse with semi-major axis $a > 0$ and semi-minor axis $b > 0$ is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Find $f(x, y)$ if each level curve $f(x, y) = C$ is an ellipse centered at the origin with
   (a) semi-major axis $\sqrt{2C}$ and semi-minor axis $\sqrt{C}$;
   (b) semi-major axis and semi-minor axis both equal to $\sqrt{\ln(2C)}$;
   (c) semi-major axis C and semi-minor axis $\sqrt{C}$.

6. Evaluate the limit
$$\lim_{x\to 0^+} \frac{\ln(\sin(x))}{e^{\frac{1}{x}}}$$

7. Find the area of the triangle with vertices $(3, 1, 2)$, $(1, 2, 2)$ and $(4, 0, 0)$.

8. Consider the function $f(x) = 2x^2 - 4x + 6$ with domain $(-\infty, 0]$, which is invertible. Give the inverse function. Also give the domain of this inverse function.

Open questions

The next questions need to be worked out completely, every answer needs to be reasoned.

9. Consider a function $f(x, y)$ which satisfies $f(1, 2) = 4$. Suppose a normal to the tangent plane of f at the point $(x, y) = (1, 2)$ is given by the vector $\langle 4, -6, -2 \rangle$. What are $f_x(1, 2)$ and $f_y(1, 2)$?

10. Evaluate the integral $\displaystyle\int \frac{1}{e^x + 3 + 2e^{-x}}\,dx$.

11. Find all points on the surface with equation $z = x^2 - 4xy^2 + 2xy$ where the tangent plane is parallel to the plane $z = 2x - 12y + 3$.

12. Determine whether the following integral is convergent or divergent. If the integral converges, you do not need to evaluate it.

$$\int_0^\infty \frac{1}{2x^{\frac{1}{3}} + 3x^3}\, dx.$$

13. Simplify $\arccos(2x) + \arcsin(2x)$ into an expression that does not involve trigonometric functions or their inverses.

THE END