Practice Exam 1 - Solutions - EE1M1 Calculus

You are allowed to use:

- Pen, pencils and scrap paper.

The formula sheet can be found on the next page.

Formula sheet

Some trigonometric formulae

$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$
$\cos(2\alpha) = 2\cos^2(\alpha) - 1 = 1 - 2\sin^2(\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$

Some limits

$$\lim_{x\to\infty} \frac{x^p}{e^x} = 0$$
$$\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x = e^a$$
$$\lim_{x\to\infty} \frac{\ln x}{x^p} = 0 \quad (p > 0)$$

Some integrals

$$\int \frac{\mathrm{d}\,x}{\sin(x)} = \ln\left|\tan\left(\frac{x}{2}\right)\right| + C$$
$$\int \frac{\mathrm{d}\,x}{\cos(x)} = \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$$
$$\int \frac{\mathrm{d}\,x}{1+x^2} = \arctan(x) + C$$
$$\int \frac{\mathrm{d}\,x}{1-x^2} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C$$
$$\int \frac{\mathrm{d}\,x}{\sqrt{1-x^2}} = \arcsin(x) + C = -\arccos(x) + C$$
$$\int \frac{\mathrm{d}\,x}{\sqrt{x^2+1}} = \ln\left(x + \sqrt{x^2+1}\right) + C$$
$$\int \frac{\mathrm{d}\,x}{\sqrt{x^2-1}} = \ln\left|x + \sqrt{x^2-1}\right| + C$$
$$\int \sqrt{1+x^2}\ \mathrm{d}\,x = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\ln\left(x + \sqrt{1+x^2}\right) + C$$
$$\int \sqrt{1-x^2}\ \mathrm{d}\,x = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin(x) + C$$
$$\int_0^{\frac{\pi}{2}} \sin^n x\ \mathrm{d}\,x = \begin{cases} \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\cdots\frac{3}{4}\frac{1}{2}\frac{\pi}{2} & \text{if } n \text{ even and } n \geq 2 \\ \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\cdots\frac{4}{5}\frac{2}{3} & \text{if } n \text{ odd and } n \geq 3 \end{cases}$$

Short-answer questions
An explanation is not required for the short-answer questions. Only the answer matters. You do not need to fully simplify your answers.

1. Find all horizontal and vertical asymptotes of the function $f(x) = \dfrac{(x-5)\sqrt{x^2+1}+2x-10}{x^2-2x-15}$.

   A correct solution is: We first simplify the function by dividing out the common factor $x-5$, which gives $f(x) = \dfrac{\sqrt{x^2+1}+2}{x+3}$. At $x=-3$, the denominator is zero, while the numerator is nonzero, so f has a vertical asymptote at $x=-3$. By applying the square root trick we can write $f(x) = \dfrac{3-x^2}{(x+3)(2-\sqrt{x^2+1})}$. We can now comput by dividing by the dominant term x^2 and taking this inside the square root

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\frac{3}{x^2}-1}{\frac{2}{x}+\frac{6}{x^2}-\sqrt{1+\frac{1}{x^2}}-\frac{3}{x}\sqrt{1+\frac{1}{x^2}}} = 1.$$

   For the limit as $x \to -\infty$, we also divide by x^2, but in order to take this term inside the square root, we have to use that $\frac{1}{x} = -\frac{1}{-x} = -\frac{1}{\sqrt{x^2}}$ since x is negative. This yields

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\frac{3}{x^2}-1}{\frac{2}{x}+\frac{6}{x^2}+\sqrt{1+\frac{1}{x^2}}+\frac{3}{x}\sqrt{1+\frac{1}{x^2}}} = -1.$$

   So f has a vertical asymptote at $x=-3$ and horizonatal asymptotes at $y=1$ and $y=-1$.

2. Consider $f(x) = \sqrt{x}$. Give the sharpest upper bound for the Lagrange remainder when approximating $\sqrt{16.02}$ using a Taylor polynomial of degree 3 with center 16.

   A correct solution is: In order to bound the Lagrange remainder, we evaluate

$$f^{(4)}(x) = -\frac{15}{16x^{\frac{7}{2}}}$$

   For s in between 16 and 16.02, the value $|f^{(4)}(s)|$ is maximal when $s=16$, in which case $|f^{(4)}(16)| = \frac{15}{262144}$. As such, the Lagrange remainder E_3 can be bounded by

$$E_3 \leq \frac{1}{4!}\frac{15}{262144}|16.02-16|^4 = \frac{1}{2621440000000}$$

3. Evaluate the integral

$$\int \arcsin(2x)\, dx.$$

   A correct solution is: We use integration by parts to obtain

$$\begin{aligned}\int \arcsin(2x)\, dx &= \int 1 \cdot \arcsin(2x)\, dx \\ &= x\arcsin(2x) - \int x\frac{2}{\sqrt{1-4x^2}}\, dx\end{aligned}$$

For the final integral, we use the substitution $u = 1 - 4x^2$. With $du = -8xdx$, we obtain

$$\begin{aligned}
\int \arcsin(2x)\, dx &= x\arcsin(2x) - \int x\frac{2}{\sqrt{1-4x^2}}\, dx \\
&= x\arcsin(2x) - \int \frac{-1}{4\sqrt{u}}\, dx \\
&= x\arcsin(2x) + \frac{1}{2}\sqrt{u} + C \\
&= x\arcsin(2x) + \frac{1}{2}\sqrt{1-4x^2} + C
\end{aligned}$$

4. Find an equation of the tangent line to the curve

$$x^3 - \cos\left(\frac{2\pi x}{y}\right) = yx^2 + 3$$

at the point $(2, 1)$.

A correct solution is: We differentiate this equation implicitly to obtain

$$3x^2 + \sin\left(\frac{2\pi x}{y}\right)\left(\frac{2\pi}{y} - \frac{2\pi x}{y^2}\frac{dy}{dx}\right) = x^2\frac{dy}{dx} + 2xy.$$

In this we can plug in $x = 2$ and $y = 1$ to obtain

$$12 + \sin(4\pi)(2\pi - 4\pi\frac{dy}{dx}) = 4\frac{dy}{dx} + 4,$$

which implies

$$12 = 4\frac{dy}{dx} + 4.$$

This means that $\frac{dy}{dx}$ at $(x, y) = (2, 1)$ equals 2. Therefore, the equation of the tangent line is given by

$$y - 1 = 2(x - 2) \text{ or } y = 2x - 3.$$

5. Recall that the equation of an ellipse with semi-major axis $a > 0$ and semi-minor axis $b > 0$ is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Find $f(x, y)$ if each level curve $f(x, y) = C$ is an ellipse centered at the origin with

(a) semi-major axis $\sqrt{2C}$ and semi-minor axis $\sqrt{C}$;

(b) semi-major axis and semi-minor axis both equal to $\sqrt{\ln(2C)}$;

(c) semi-major axis C and semi-minor axis $\sqrt{C}$.

A correct solution is:

(a) We can rewrite the equation $\frac{x^2}{2C} + \frac{y^2}{C} = 1$ to $\frac{x^2}{2} + y^2 = C$, which yields $f(x, y) = \frac{x^2}{2} + y^2$.

(b) We can rewrite the equation $\frac{x^2}{\ln(2C)} + \frac{y^2}{\ln(2C)} = 1$ to $x^2 + y^2 = \ln(2C)$, from which we deduce $2C = e^{x^2+y^2}$, which yields $f(x, y) = \frac{1}{2}e^{x^2+y^2}$.

(c) We can rewrite the equation $\frac{x^2}{C^2} + \frac{y^2}{C} = 1$ to $x^2 + y^2C = C^2$, which we can solve for C to obtain $C = \frac{1}{2}(y^2 - \sqrt{4x^2 + y^4})$ or $C = \frac{1}{2}(y^2 + \sqrt{4x^2 + y^4})$. However, in the first case C will always turn out negative, which is not possible since C is the semi-major axis and therefore nonnegative. As such, $f(x, y) = \frac{1}{2}(y^2 + \sqrt{4x^2 + y^4})$.

6. Evaluate the limit

$$\lim_{x\to 0^+} \frac{\ln(\sin(x))}{e^{\frac{1}{x}}}$$

A correct solution is: Since both the numerator and the denominator of the fraction approach either ∞ or $-\infty$ as $x \to 0^+$, we can apply l'Hospital's rule. This yields

$$\lim_{x\to 0^+} \frac{\ln(\sin(x))}{e^{\frac{1}{x}}} = \lim_{x\to 0^+} \frac{\frac{\cos(x)}{\sin(x)}}{-\frac{1}{x^2}e^{\frac{1}{x}}} = \lim_{x\to 0^+} \frac{\frac{-\cos(x)x^2}{\sin(x)}}{e^{\frac{1}{x}}}.$$

Now, either using a standard limit or again using l'Hospital's rule, we find $\lim_{x\to 0^+} \dfrac{x}{\sin(x)} = 1$. This means that $\lim_{x\to 0^+} \dfrac{-\cos(x)x^2}{\sin(x)} = 0$. Since the denominator $e^{\frac{1}{x}}$ approaches ∞ as $x \to 0^+$, we find that

$$\lim_{x\to 0^+} \frac{\ln(\sin(x))}{e^{\frac{1}{x}}} = \lim_{x\to 0^+} \frac{\frac{-\cos(x)x^2}{\sin(x)}}{e^{\frac{1}{x}}} = 0.$$

7. Find the area of the triangle with vertices $(3, 1, 2)$, $(1, 2, 2)$ and $(4, 0, 0)$.

A correct solution is: The triangle is spanned by the vectors $\mathbf{u} = \langle -2, 1, 0\rangle$ and $\mathbf{v} = \langle 1, -1, -2\rangle$, i.e. the vectors from $(3, 1, 2)$ to $(1, 2, 2)$ and from $(3, 1, 2)$ to $(4, 0, 0)$ respectively. The norm $||\mathbf{u}\times\mathbf{v}||$ represents the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$, so $\frac{1}{2}||\mathbf{u}\times\mathbf{v}||$ represents the area of the given triangle. By computing $\mathbf{u}\times\mathbf{v} = \langle -2, -4, 1\rangle$, we find that the area is given by $\frac{1}{2}||\mathbf{u} \times \mathbf{v}|| = \frac{1}{2}\sqrt{(-2)^2 + (-4)^2 + 1^2} = \frac{1}{2}\sqrt{21}$.

8. Consider the function $f(x) = 2x^2 - 4x + 6$ with domain $(-\infty, 0]$, which is invertible. Give the inverse function. Also give the domain of this inverse.
A correct solution is: We write

$$y = 2x^2 - 4x + 6$$

and solve for x, which gives

$$x = 1 \pm \frac{\sqrt{y-4}}{\sqrt{2}}$$

Since the domain of f is $(-\infty, 1]$, and the domain of f is the range of f^{-1}, we should take the negative square root. Indeed, we find that $f^{-1}(x) = 1 - \frac{\sqrt{x-4}}{\sqrt{2}}$. The domain of f^{-1} equals the range of f. Clearly $\lim_{x\to-\infty} f(x) = \infty$. Moreover, f is decreasing on its domain. So the range of f is $[f(0), \infty) = [6, \infty)$, which must therefore also be the domain of f^{-1}.

Open questions

The next questions need to be worked out completely, every answer needs to be reasoned.

9. Consider a function $f(x, y)$ which satisfies $f(1, 2) = 4$. Suppose a normal to the tangent plane of f at the point $(x, y) = (1, 2)$ is given by the vector $\langle 4, -6, -2\rangle$. What are $f_x(1, 2)$ and $f_y(1, 2)$?

A correct solution is: On the one hand, an equation for the tangent plane is given by

$$z = 4 + f_x(1,2)(x-1) + f_y(1,2)(y-2),$$

while, using the normal, we find that an equation for the tangent plane is given by

$$4(x-1) - 6(y-2) - 2(z-4) = 0,$$

i.e. $2z = 8 + 4(x-1) - 6(y-2)$. We can divide this equation by 2 to obtain $z = 4 + 2(x-1) - 3(y-2)$. Now we have two equations for the same tangent plane, so we can read off that $f_x(1,2) = 2$ and $f_y(1,2) = -3$.

10. Evaluate the integral $\displaystyle\int \frac{1}{e^x + 3 + 2e^{-x}}\,dx$.

A correct solution is: We will first multiply both the numerator and the denominator by e^x. Then we use the substitution $u = e^x$ with $du = e^x dx$. This yields

$$\int \frac{1}{e^x + 3 + 2e^{-x}}\,dx = \int \frac{e^x}{e^{2x} + 3e^x + 2}\,dx = \int \frac{1}{u^2 + 3u + 2}\,du.$$

Since $u^2 + 3u + 2 = (u+2)(u+1)$ we use partial fraction decomposition. This gives

$$\int \frac{1}{u^2 + 3u + 2}\,du = \int \frac{1}{u+1} - \frac{1}{u+2}\,du = \ln(|u+1|) - \ln(|u+2|) + C.$$

Plugging back in $u = e^x$ we find

$$\int \frac{1}{e^x + 3 + 2e^{-x}}\,dx = \ln(e^x + 1) - \ln(e^x + 2) + C.$$

11. Find all points on the surface with equation $z = x^2 - 4xy^2 + 2xy$ where the tangent plane is parallel to the plane $z = 2x - 12y + 3$.

A correct solution is: We first compute the equation for the tangent plane at an arbitrary point (x_0, y_0). The partial derivatives are given by

$$\frac{\partial}{\partial x}[x^2 - 4xy^2 + 2xy] = 2x - 4y^2 + 2y, \quad \frac{\partial}{\partial y}[x^2 - 4xy^2 + 2xy] = -8xy + 2x.$$

At (x_0, y_0) the tangent plane is given by the equation

$$z = x_0^2 - 4x_0y_0^2 + 2x_0y_0 + (2x_0 - 4y_0^2 + 2y_0)(x - x_0) + (-8x_0y_0 + 2x_0)(y - y_0).$$

Recall that two parallel planes have the same normal vectors. The tangent plane has normal vector $\langle -(2x_0 - 4y_0^2 + 2y_0), -(-8x_0y_0 + 2x_0), 1\rangle$, while the plane $z = 2x - 12y + 3$ has normal vector $\langle -2, 12, 1\rangle$. For the tangent plane to be parallel to $z = 2x - 12y + 3$, we need to solve the system

$$\begin{cases} 2x_0 - 4y_0^2 + 2y_0 = 2, \\ -8x_0y_0 + 2x_0 = -12. \end{cases}$$

From the first equation we can derive $x_0 = 1 + 2y_0^2 - y_0$. Substituting this in the second equation yields $-16y_0^3 + 12y_0^2 - 10y_0 + 14 = 0$. $y_0 = 1$ can easily seen to be a solution of this. Performing a long division, we obtain the solution $y_0 = 1$ or $-16y_0^2 - 4y_0 - 14 = 0$, of which the latter equation does not have a solution. Hence, we find $x_0 = 2$, which yields that $(2, 1)$ is the only point at which the tangent plane is parallel to $z = 2x - 12y + 3$.

12. Determine whether the following integral is convergent or divergent. If the integral converges, you do not need to evaluate it.

$$\int_0^\infty \frac{1}{2x^{\frac{1}{3}} + 3x^3}\, dx.$$

A correct solution is: This integral is improper of both type 1 and type 2, so we split the integral as

$$\int_0^\infty \frac{1}{2x^{\frac{1}{3}} + 3x^3}\, dx = \int_0^1 \frac{1}{2x^{\frac{1}{3}} + 3x^3}\, dx + \int_1^\infty \frac{1}{2x^{\frac{1}{3}} + 3x^3}\, dx.$$

For the first integral, we note that $\frac{1}{2x^{\frac{1}{3}}+3x^3} \leq \frac{1}{2x^{\frac{1}{3}}}$ for $0 < x < 1$. Since the integral $\int_0^1 \frac{1}{2x^{\frac{1}{3}}}\, dx$ converges, the integral $\int_0^1 \frac{1}{2x^{\frac{1}{3}}+3x^3}\, dx$ converges as well. For the second integral, we note that $\frac{1}{2x^{\frac{1}{3}}+3x^3} \leq \frac{1}{3x^3}$ for $x > 1$. Since the integral $\int_1^\infty \frac{1}{3x^3}\, dx$ converges, the integral $\int_1^\infty \frac{1}{2x^{\frac{1}{3}}+3x^3}\, dx$ converges as well. Therefore, the improper integral converges.

13. Simplify $\arccos(2x) + \arcsin(2x)$ into an expression that does not involve trigonometric functions or their inverses.

A correct solution is: Consider the function $f(x) = \arccos(2x) + \arcsin(2x)$. Then we can compute

$$f'(x) = -\frac{2}{\sqrt{1-4x^2}} + \frac{2}{\sqrt{1-4x^2}} = 0.$$

This means that the function f is constant, since its derivative is 0. In order to determine which constant, we plug in any value of x for which we can compute $f(x)$ by hand. In this case, $x = 0$ will do nicely. In particular, we find that

$$\arccos(2x) + \arcsin(2x) = f(x) = f(0) = \arccos(0) + \arcsin(0) = \frac{\pi}{2} + 0 = \frac{\pi}{2}$$

for any value of x for which the expression is defined.

THE END