EE1C1 "Linear Circuits A"

Week 1.7

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Today

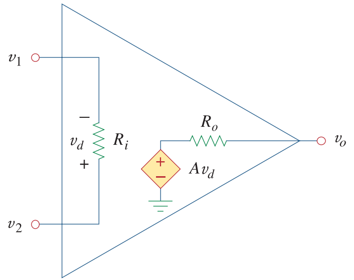
- Recap week 1.6:
  - Circuits with operational amplifiers
  - How to solve circuits with op-amp -> simplifying assumptions for ideal op-amp
  - Inverting & Non-inverting configurations

- Week 1.7:
  - Capacitance / Capacitor
  - Inductance / Inductor
  - C–C and L–L interconnections

- Summary and Next Week

Recap of week 1.6

- Operational amplifiers:
  - ideal: $A \to \infty$; $R_\mathrm{i} \to \infty$; $R_\mathrm{o} = 0$
  - non-ideal



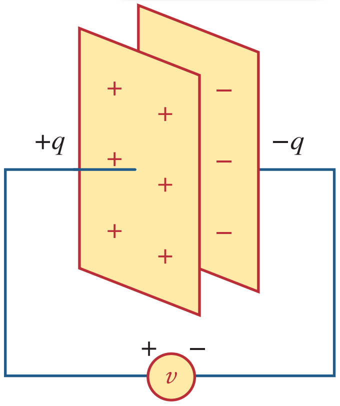
- Ideal Op-Amp is your friend in a circuit:
  - currents into both input terminals are zero
  - under feedback operation, the two inputs have the same voltage

Week 1.7

The capacitance and the capacitor

Capacitance

Capacitance & capacitor

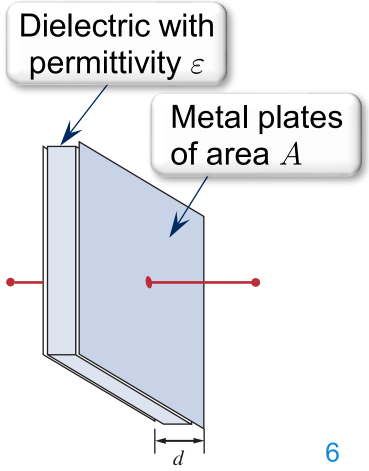


- Capacitor: a passive element designed to store energy in its electric field

- In a linear capacitor: $q = Cv$ with C = capacitance, measure unit farad (F)

- Typical example: the parallel-plate capacitor

$$C = \frac{\varepsilon A}{d} = \frac{\varepsilon_\mathrm{r}\varepsilon_0\, A}{d}$$



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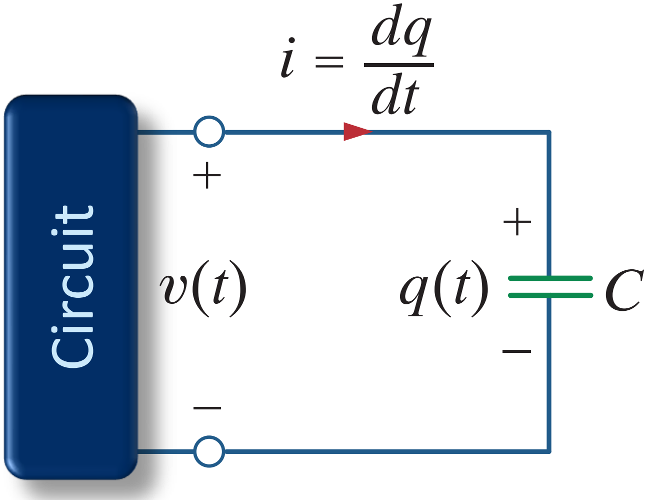
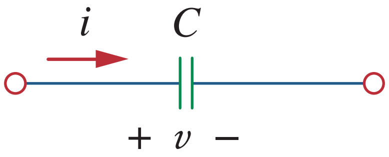
Capacitance & capacitor

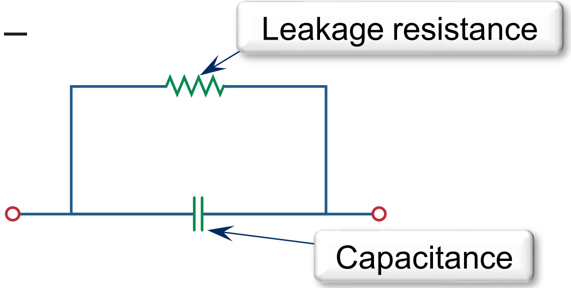
- Circuital perspective →



- Ideally: capacitors have only capacitance → as circuit elements, we refer to them as capacitances
- Symbol:



- Real capacitors:



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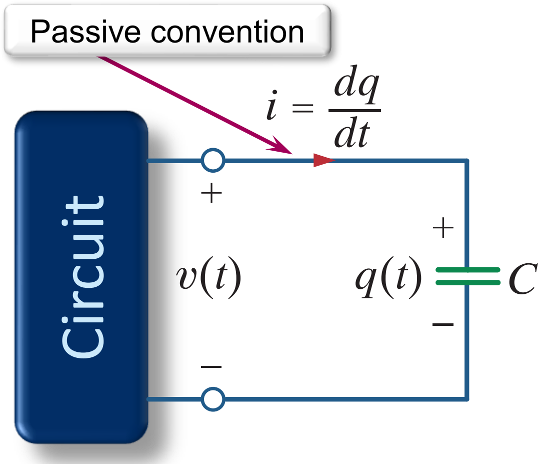
Capacitance: features

Basic relation: $v \to i$



$$i(t) = \frac{\mathrm{d}q(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}[Cv(t)] = C\frac{\mathrm{d}v(t)}{\mathrm{d}t}$$

$$i(t) = C\frac{\mathrm{d}v(t)}{\mathrm{d}t}$$

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\mathrm{d}\tau = \underbrace{\frac{1}{C}\int_{-\infty}^{t_0} i(\tau)\mathrm{d}\tau}_{\text{the voltage at } t_0} + \frac{1}{C}\int_{t_0}^{t} i(\tau)\mathrm{d}\tau$$

Reciprocal relation: $i \to v$

$$v(t) = v(t_0) + \frac{1}{C}\int_{t_0}^{t} i(\tau)\mathrm{d}\tau$$

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Capacitance: features

$$\text{energy} = \int \text{power}\,\mathrm{d}t$$

- Capacitances only store or release electrostatic energy → they do not generate any energy
- Power: $p(t) = v(t)i(t) = Cv(t)\dfrac{\mathrm{d}v(t)}{\mathrm{d}t}$ (watt;W)
- Energy:

$$w(t) = \int_{-\infty}^{t} p(\tau)\mathrm{d}\tau = C\int_{-\infty}^{t} v(\tau)\frac{\mathrm{d}v}{\mathrm{d}\tau}\mathrm{d}\tau = C\int_{v(-\infty)}^{v(t)} v\mathrm{d}v = \frac{1}{2}Cv^2\Big|_{v(-\infty)}^{v(t)}$$

$$w(t) = \frac{1}{2}Cv^2(t) \text{ (joule;J)}$$

$v(-\infty) = 0$

Assuming that the capacitor was totally discharged at $t = -\infty$

Capacitance: features

energy $= \int$ power $\mathrm{d}t$

- Capacitances only store or release electrostatic energy → they do not generate any energy

- Energy – there are 3 variants basically
  - from an initial 0-energy to a well-define state at t: $w(t) = \frac{1}{2} C v^2(t)$
  - accounting for an initial energy: $w = C \int_{v(t_0)}^{v(t)} v \frac{\mathrm{d}v}{\mathrm{d}\tau} \mathrm{d}\tau + w(t_0)$
  - energy stored over an interval: $\Delta w_{12} = C \int_{v(t_1)}^{v(t_2)} v \frac{\mathrm{d}v}{\mathrm{d}\tau} \mathrm{d}\tau$

Capacitance: features

$$\text{energy} = \int \text{power}\, \mathrm{d}t$$

- Capacitances only store or release electrostatic energy → they do not generate any energy

- Energy:
  - from an initial 0-energy to a well-define state at t: $w(t) = \frac{1}{2}Cv^2(t)$
  - substituting water molecules for electric charges → it looks like "filling" an empty glass with water

    see Ali Sheikholeslami, "A Capacitor Analogy," parts 1 & 2 on BS

Capacitance: features

- Consequence of the integral-form:

$$v(t) = v(t_0) + \frac{1}{C}\int_{t_0}^{t} i(\tau)\mathrm{d}\tau$$

- **The capacitance voltage is continuous**

- Consequence of the differential form:

$$i(t) = C\frac{\mathrm{d}v(t)}{\mathrm{d}t} = 0$$

- **In steady state** ($v(t) =$ constant)

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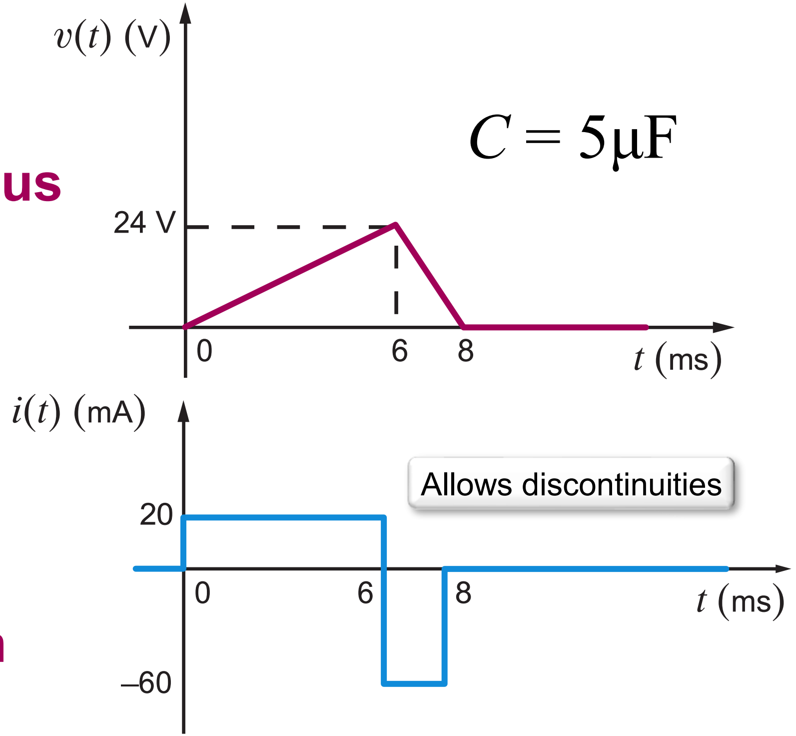
  the capacitance behaves as an open circuit



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Capacitance: example

The current through a 0.5 F capacitor is $i(t) = 6(1 - e^{-t})A$

Calculate the voltage and power at t=2s. Assume v(0)=0V.

To find the **voltage** apply the equations seen a few slides before.

$$v(t) = v(0) + \frac{1}{C}\int_0^t i(\tau)d\tau$$

In this case this means $v(t) = \frac{1}{0.5}\int_0^t 6(1 - e^{-\tau})d\tau$ - Solve the integral and calculate the voltage at t=2s. [Expected result $\mathrm{V(t)} = 12(t + e^{-t}) - 12\ V$]

To find the **power**, you can simply multiply the voltage and the current and calculate the value at t=2s. [Expected result P=70.6W]

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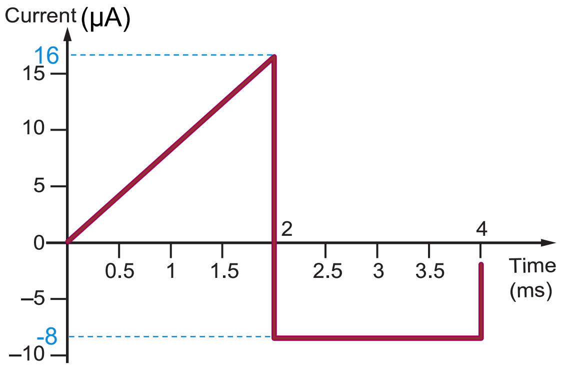
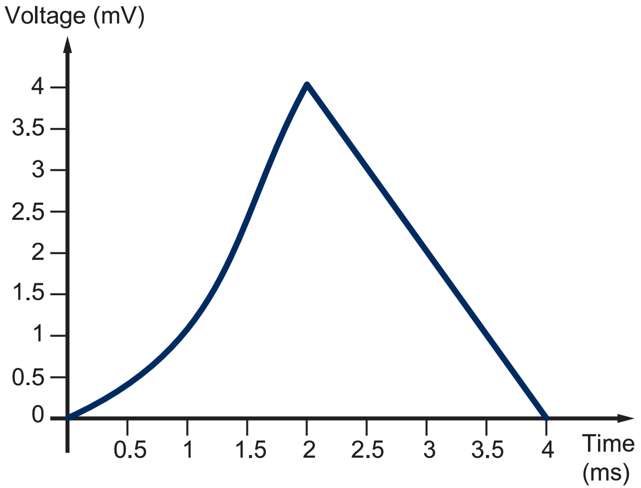
Capacitance: example 2

- For the given current, calculate the **voltage** on the 4μF capacitance, by accounting for $v(0) = 0$.



$$v(t) = v(t_0) + \frac{1}{C}\int_{t_0}^{t} i(\tau)\mathrm{d}\tau$$



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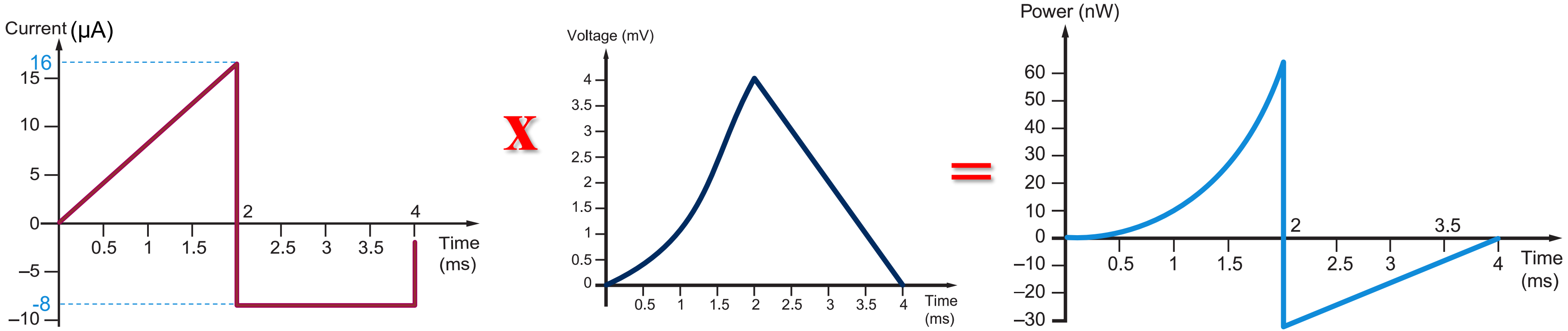
Capacitance: example 2

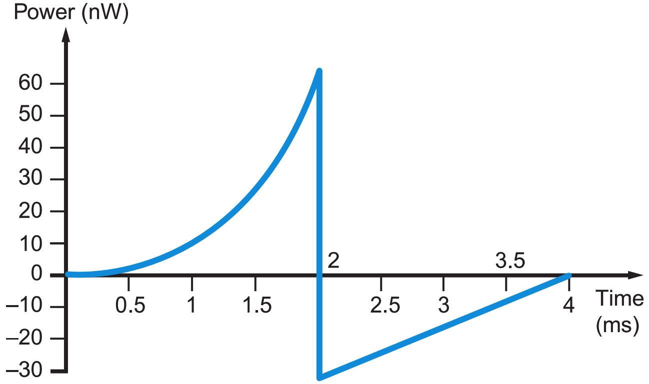
- Calculate the **power** flow due to the 4μF capacitance.

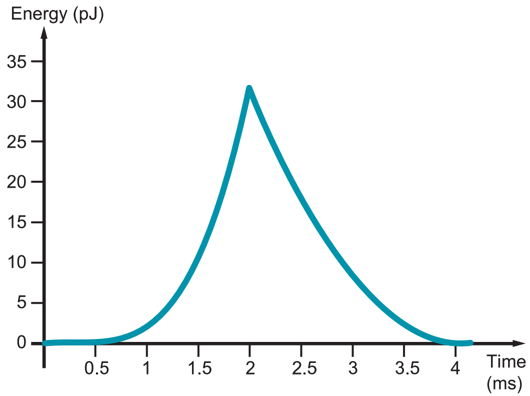


$$p(t) = v(t)i(t)$$

Capacitance: example 2

- Calculate the **energy** stored in the 4μF capacitance.





$$w(t) = C\int_{-\infty}^{t} v(\tau)\frac{\mathrm{d}v}{\mathrm{d}\tau}\mathrm{d}\tau \qquad w(t) = \int_{-\infty}^{t} p(\tau)\mathrm{d}\tau$$

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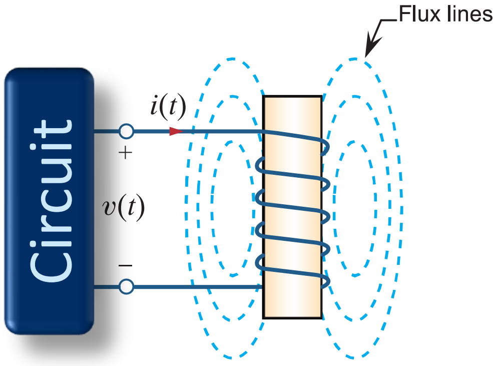
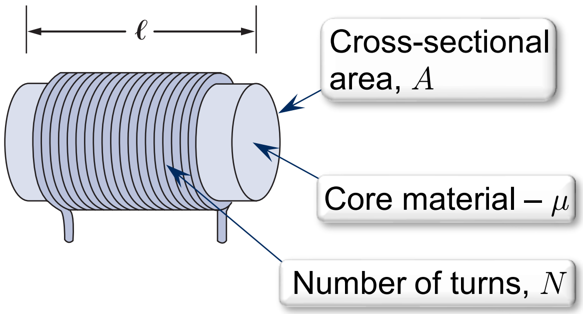
The inductance and the inductor

Inductance & inductor



- Inductor: a passive element designed to store energy in its magnetic field
- In a linear inductor: $v(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t}$ with L = inductance, measure unit henry (H)



- Typical example: the solenoid

$$L = \frac{\mu N^2 A}{\ell} = \frac{\mu_\mathrm{r}\mu_0 N^2 A}{\ell}$$

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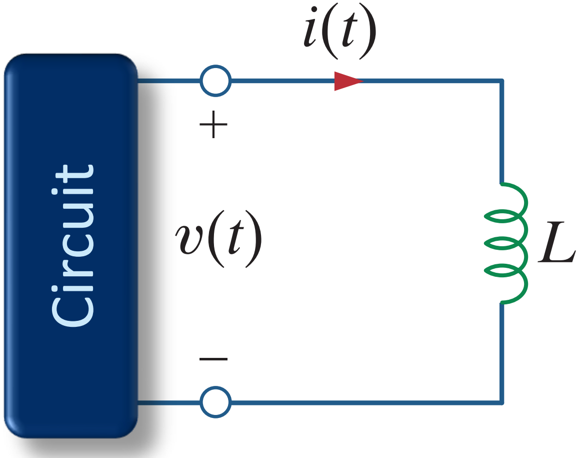
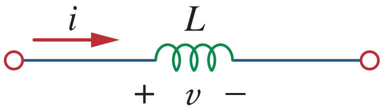
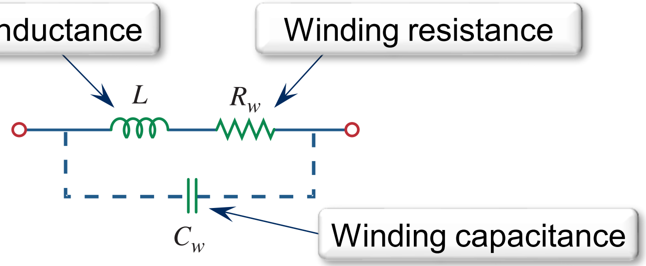
Inductance & inductor

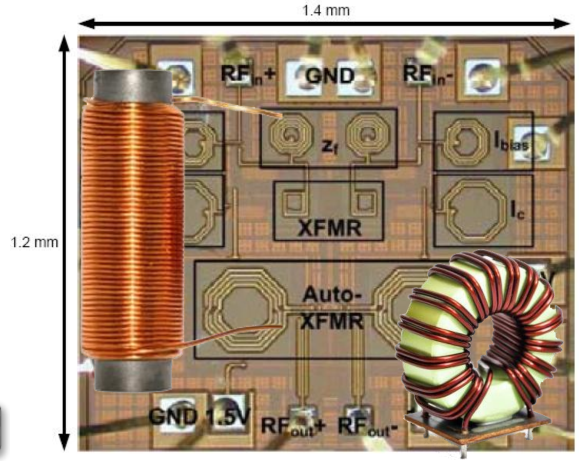


- Circuital perspective
- Ideally: inductors have only inductance as circuit elements, we refer to them as inductances
- Symbol:



- Real inductors:





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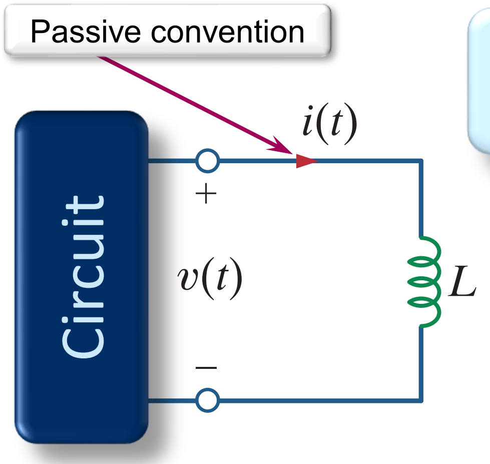
Inductance: features



To be explained in "Electricity & Magnetism"

Basic relation: $i \rightarrow v$

$$\Phi(t) = L\, i(t) \rightarrow v(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t}$$

$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)\mathrm{d}\tau = \underbrace{\frac{1}{L}\int_{-\infty}^{t_0} v(\tau)\mathrm{d}\tau}_{\text{the current at } t_0} + \frac{1}{L}\int_{t_0}^{t} v(\tau)\mathrm{d}\tau$$

Reciprocal relation: $v \rightarrow i$

$$i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} v(\tau)\mathrm{d}\tau$$

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Inductance: features

$$\text{energy} = \int \text{power}\,\mathrm{d}t$$

- Inductances only store or release magnetic energy → they do not generate any energy
- Power: $p(t) = i(t)v(t) = Li(t)\frac{\mathrm{d}i(t)}{\mathrm{d}t}$ (watt;W)
- Energy:

$$w(t) = \int_{-\infty}^{t} p(\tau)\mathrm{d}\tau = L\int_{-\infty}^{t} i(\tau)\frac{\mathrm{d}i}{\mathrm{d}\tau}\mathrm{d}\tau = L\int_{i(-\infty)}^{i(t)} i\mathrm{d}i = \frac{1}{2}Li^2\Big|_{i(-\infty)}^{i(t)}$$

$i(-\infty) = 0$

$$w(t) = \frac{1}{2}Li^2(t) \text{ (joule;J)}$$

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Inductance: features

$$\text{energy} = \int \text{power}\,\mathrm{d}t$$

- Inductances only store or release magnetic energy → they do not generate any energy

- Energy:
  - from an initial 0-energy to a well-define state at t: $w(t) = \frac{1}{2} L i^2(t)$
  - accounting for an initial energy: $w = L \int_{i(t_0)}^{i(t)} i \frac{\mathrm{d}i}{\mathrm{d}\tau} \mathrm{d}\tau + w(t_0)$
  - energy stored over an interval: $\Delta w_{12} = L \int_{i(t_1)}^{i(t_2)} i \frac{\mathrm{d}i}{\mathrm{d}\tau} \mathrm{d}\tau$

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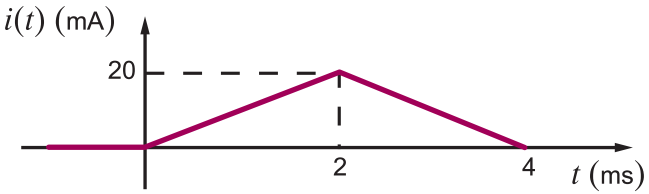
Inductance: features

- Consequence of the integral-form:

$$i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} v(\tau)\mathrm{d}\tau$$

- The inductance current is continuous



- Consequence of the differential form:

$$v(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t} = 0$$

- In steady state ($i(t) =$ constant)

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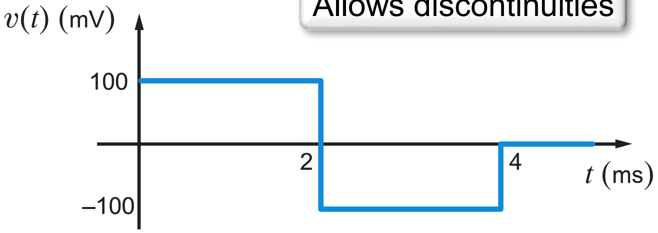
  the inductance behaves as a short-circuit

Allows discontinuities



Inductance: example

Consider a 0.5 H inductance and a current flowing through it expressed as below:

$$i(t) = \begin{cases} 0\ A & t \leq 0 \\ 2te^{-4t} A & t > 0 \end{cases}$$

Calculate the voltage across the inductance and the power at t=1s.

-Recall that V(t)=LdI(t)/dt…so simply apply the formula.

-Then simply use the general relation between instantaneous power, voltage and current.

Expected result $\mathrm{V(t)} = e^{-4t}(1-4t)V \text{ and } -6e^{-8}W$

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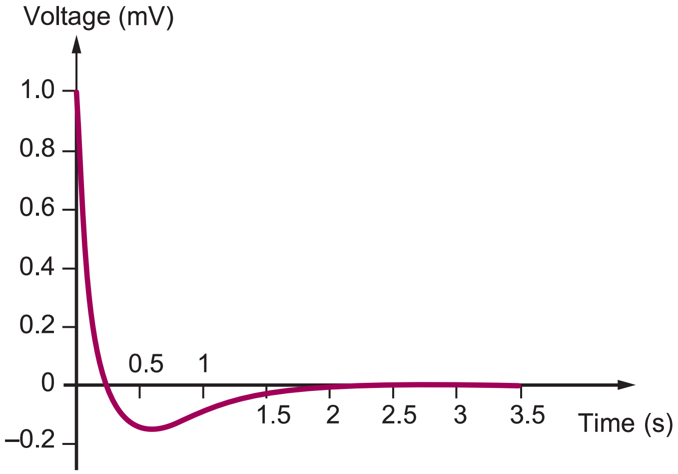
Inductance: example 2

- The voltage over a 200mH inductance can be expressed as: Calculate the current, the power and the energy.

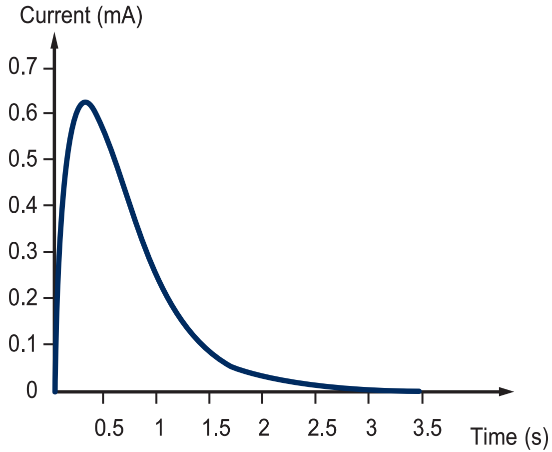
$$v(t) = \begin{cases} (1-3t)\exp(-3t) \text{ mV}, \ t \geq 0 \\ 0, \ t<0 \end{cases}$$



Inductance: example 2

- The voltage over a 200mH inductance can be expressed as: Calculate the current, the power and the energy.

$$v(t)=\begin{cases}(1-3t)\exp(-3t)\ \text{mV},\ t\geq 0\\ 0,\ t<0\end{cases}$$



$$i(t)=i(0)+\frac{1}{L}\int_0^t v(\tau)d\tau$$

$$=0+\frac{10^3}{200}\int_0^t (1-3\tau)\exp(-3\tau)d\tau$$

$$=\frac{10^3}{200}\left(\int_0^t (\exp(-3\tau)d\tau-3\int_0^t \tau\exp(-3\tau)d\tau\right)$$

$$=\frac{10^3}{200}\left(\left[\frac{\exp(-3\tau)}{-3}\right]_0^t-3\left[\frac{\tau\exp(-3\tau)}{-3}+\frac{\exp(-3\tau)}{-9}\right]_0^t\right)$$

$$=5t\exp(-3t)\ \text{mA},\ t\geq 0$$

Integration by parts -> $\int f(\tau)g'(\tau)d\tau = f(\tau)g(\tau)-\int f'(\tau)g(\tau)d\tau$

Inductance: example 2

- The voltage over a 200mH inductance can be expressed as: $v(t) = \begin{cases} (1-3t)\exp(-3t)\ \text{mV},\ t \geq 0 \\ 0,\ \text{t<0} \end{cases}$
  Calculate the current, the power and the energy.

$$p(t) = v(t)i(t) = 5\times 10^{-6} \cdot t(1-3t)\exp(-6t)\ \text{W}$$

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Inductance: example 2

- The voltage over a 200mH inductance can be expressed as: $v(t)=\begin{cases}(1-3t)\exp(-3t)\ \text{mV},\ t\geq 0\\ 0,\ t<0\end{cases}$
  Calculate the current, the power and the energy.

$$w(t)=\frac{1}{2}Li^2(t)=\frac{1}{2}\cdot 200\times 10^{-3}\cdot\left(\frac{1}{200}t\exp(-3t)\right)^2=2,5\times 10^{-6}t^2\exp(-6t)\ \text{J},\ \ t\geq 0$$

$$w(t)=\int_{-\infty}^{t}p(x)dx=5\times 10^{-6}\int_{-\infty}^{t}\tau(1-3\tau)\exp(-6\tau)d\tau=2,5\times 10^{-6}t^2\exp(-6t)\ \text{J},\ \ t\geq 0$$

Solving the integral is more complicated in this case than the operation above. However, that formula is applicable only because the initial condition was zero.

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Coffee Break

C–C and L–L interconnections

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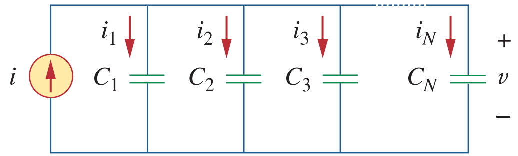
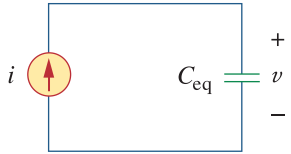
Capacitances in parallel





- Determine the equivalent capacitance in the case of several capacitances connected in parallel

- Apply KCL:

$$i = \sum_{k=1}^{N} i_k = \sum_{k=1}^{N} C_k \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}t} \sum_{k=1}^{N} C_k = C_{\mathrm{eq}} \frac{\mathrm{d}v}{\mathrm{d}t}$$

- Parallel connection: $C_{\mathrm{eq}} = \sum_{k=1}^{N} C_k$

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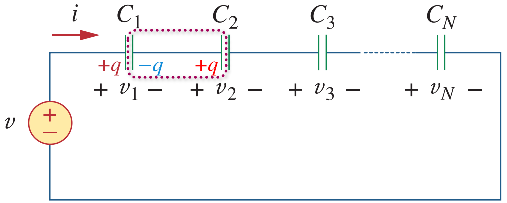
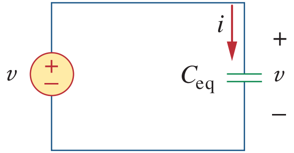
Capacitances in series



- Determine the equivalent capacitance in the case of several capacitances connected in series



- Charge distribution: $q = q_k, \;\; k = 1, \ldots, N$
- Apply KVL: $v = \sum_{k=1}^{N} v_k \longrightarrow \frac{q}{C_{eq}} = \sum_{k=1}^{N} \frac{q_k}{C_k} = q \sum_{k=1}^{N} \frac{1}{C_k}$
- Series connection: $\frac{1}{C_{eq}} = \sum_{k=1}^{N} \frac{1}{C_k}$

2 capacitances:

$$C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$$

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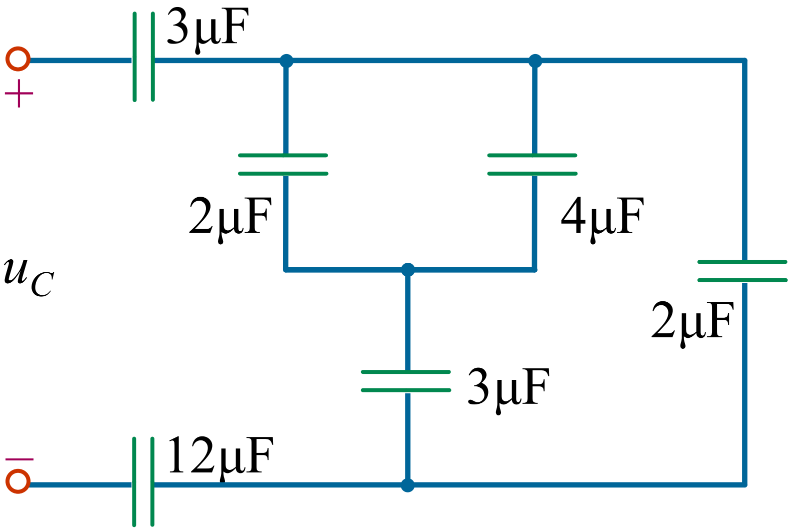
Capacitances: example

- Determine the equivalent capacitance of the circuit below



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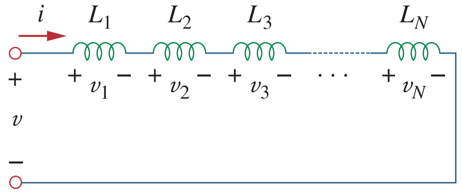
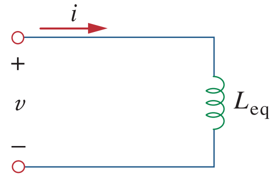
Inductances in series





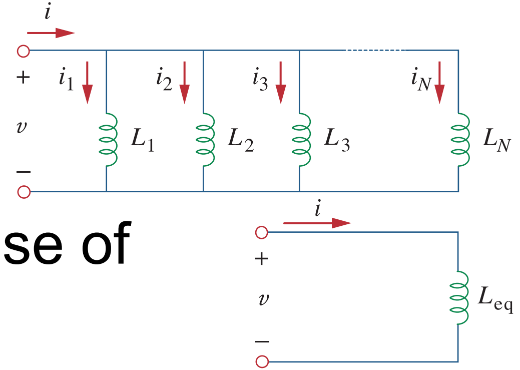
- Determine the equivalent inductance in the case of several inductances connected in series

- Apply KVL:

$$v = \sum_{k=1}^{N} v_k = \sum_{k=1}^{N} L_k \frac{\mathrm{d}i}{\mathrm{d}t} = \frac{\mathrm{d}i}{\mathrm{d}t} \sum_{k=1}^{N} L_k = L_{\mathrm{eq}} \frac{\mathrm{d}i}{\mathrm{d}t}$$

- Series connection: $$L_{\mathrm{eq}} = \sum_{k=1}^{N} L_k$$

Inductances in parallel



- Determine the equivalent inductance in the case of several inductances connected in parallel

- Apply KCL:

$$i = \sum_{k=1}^{N} i_k \;\Big|\; \frac{\mathrm{d}}{\mathrm{d}t} \longrightarrow \frac{\mathrm{d}i}{\mathrm{d}t} = \sum_{k=1}^{N} \frac{\mathrm{d}i_k}{\mathrm{d}t} = \sum_{k=1}^{N} \frac{v}{L_k} = v \sum_{k=1}^{N} \frac{1}{L_k} = v \frac{1}{L_{\mathrm{eq}}}$$

- Parallel connection:

$$\frac{1}{L_{eq}} = \sum_{k=1}^{N} \frac{1}{L_k}$$

2 inductances:

$$L_{\mathrm{eq}} = \frac{L_1 L_2}{L_1 + L_2}$$

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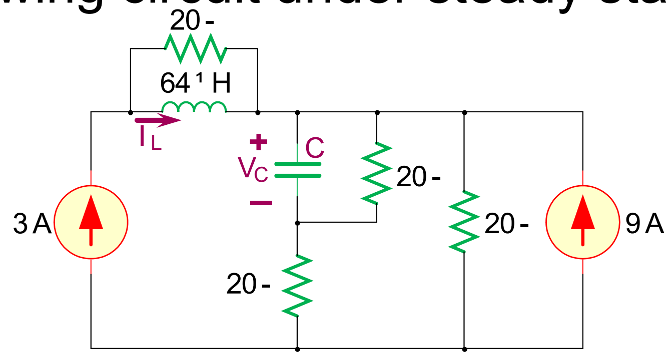
Exam exercise example

Exam(ple)

- Consider the following circuit under steady state DC conditions.



a) Calculate the voltage V_C across the capacitance and current I_L through the inductance.

b) Determine the value of the capacitance C so that the stored energy in the capacitor is equal to the stored energy in the inductor.

Note that something went wrong with the fonts: there is no minus after the values of each resistor; the inductor is measured in mH.

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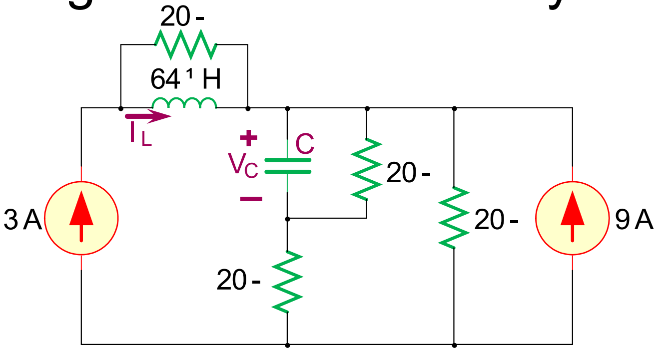
Exam(ple)

- Consider the following circuit under steady state DC conditions.

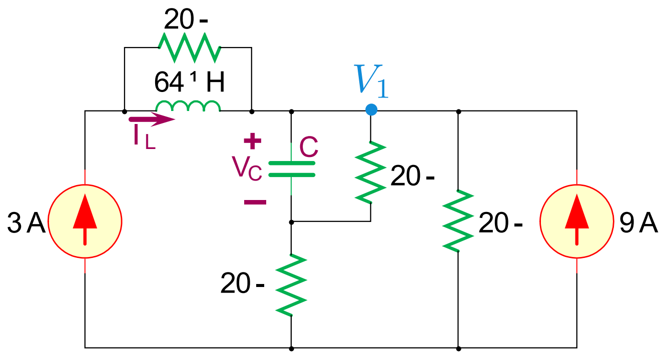


Before you start writing equations, read & look -> the 'magic words' are **under steady state DC conditions**. What happens to the capacitor and the inductor in this case?

In exam conditions I'd <u>always</u> advice you to **redraw** the circuit (even if the exercise does not explicitly ask for it).

Exam(ple)

a) Calculate the voltage V_C across the capacitance and current I_L through the inductance.



I_L ⟵ given by the 3A current source ⟶ $I_L = 3\,\mathrm{A}$

V_C ⟵ nodal analysis @ 1: $\frac{V_1}{40} + \frac{V_1}{20} = 9 + 3$ ⟶ $V_1 = 160\,\mathrm{V}$

Voltage divider on the two 20 Ohm resistors: $V_C = 80\,\mathrm{V}$

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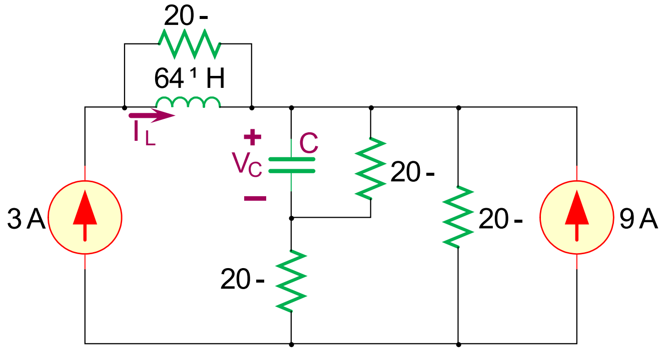
Exam(ple)

b) Determine the value of the capacitance C so that the stored energy in the capacitor is equal to the stored energy in the inductor.



- Equate the energies for L, I_L, C and V_C → $\dfrac{L\,I_L^2}{2} = \dfrac{C\,V_C^2}{2}$

- Capacitance: $C = \dfrac{LI_L^2}{V_c^2} = \dfrac{64*10^{-3}*9}{6400} = 0{,}09 * 10^{-3} = 90\mu F$

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Summary

- Capacitance (capacitors):
  - basic relations: $q = Cv$, $i = C\frac{\mathrm{d}v}{\mathrm{d}t}$, $v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)\mathrm{d}\tau + v(t_0)$
  - stored energy: $w = \frac{1}{2}Cv^2(t)$, $w = \frac{q^2}{2C}$
  - v at the capacitance terminals is continuous in t
  - parallel: $C_{\mathrm{eq}} = \sum_{k=1}^{N} C_k$ series: $\frac{1}{C_{\mathrm{eq}}} = \sum_{k=1}^{N} \frac{1}{C_k}$
- Inductance (inductors):
  - basic relations: $v = L\frac{\mathrm{d}i}{\mathrm{d}t}$, $i(t) = \frac{1}{L}\int_{t_0}^{t} v(\tau)\mathrm{d}\tau + i(t_0)$
  - stored energy: $w = \frac{1}{2}Li^2(t)$
  - i through the inductance is continuous in t
  - series: $L_{\mathrm{eq}} = \sum_{k=1}^{N} L_k$ parallel: $\frac{1}{L_{\mathrm{eq}}} = \sum_{k=1}^{N} \frac{1}{L_k}$

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Summary

TABLE 6.1

Important characteristics of the basic elements.†

Relation	Resistor (R)	Capacitor (C)	Inductor (L)
v-i:	$v = iR$	$v = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + v(t_0)$	$v = L\frac{di}{dt}$
i-v:	$i = v/R$	$i = C\frac{dv}{dt}$	$i = \frac{1}{L}\int_{t_0}^{t} v(\tau)d\tau + i(t_0)$
p or w:	$p = i^2R = \frac{v^2}{R}$	$w = \frac{1}{2}Cv^2$	$w = \frac{1}{2}Li^2$
Series:	$R_{eq} = R_1 + R_2$	$C_{eq} = \frac{C_1C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
Parallel:	$R_{eq} = \frac{R_1R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \frac{L_1L_2}{L_1 + L_2}$
At dc:	Same	Open circuit	Short circuit
Circuit variable that cannot change abruptly:	Not applicable	v	i

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Next tasks

Thank you!

My next lecture with you will be in Linear Circuits B. Success with the end-term exam and don't hesitate to contact me in case of questions!

- **SGH** (Self-Graded Homework assignments): posted today; submission due on Wednesday.
- **Seminar**: in groups on Tuesday & altogether on Friday.
- **Next week**:
  - First-order circuits with a transient (dynamic circuits)