Integrated math: Lecture 1

Linear systems and echelon forms

PRIME

TU Delft

Programme

- Course organization
- Linear systems
- Augmented matrix
- Solving linear systems
- (Reduced) echelon forms

Integrated math

- Recap important concepts and techniques from secondary school
- Introduce new material relevant for Q1 courses

Before face-to-face session

- Study the pre-lecture video

After face-to-face session

- Study the relevant sections in the book
- Do the post-lecture exercises

Assessment

- **Mandatory exercise sets**
- **Minimum number of completed exercise sets required for repair in case of missed deadline**
- **For exact rules, check BrightSpace!**

Linear systems

Linear system

Definition:
A linear equation is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

The numbers $a_1, a_2, \ldots, a_n$ are called the coefficients and the variables $x_1, x_2, \ldots, x_n$ the unknowns. The term b is referred to as the constant term.

Examples:

- $2x_1 - 3x_2 = 5$: line in x_1x_2-plane
- $3x_1 - 4x_2 + 7x_3 = 6$: plane in $x_1x_2x_3$-space

Linear system

Definition:
A set of one or more linear equations is called a system of linear equations (or linear system, for short). In the case of m linear equations in the variables $x_1, x_2, \ldots, x_n$ we speak of a system of m equations in n unknowns. The most general system then looks as follows:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad\vdots \qquad\quad \vdots \qquad\quad \ldots \qquad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Solutions of linear systems

Definition:
A solution of a linear system is an ordered list of n values $(c_1, c_2, \ldots, c_n)$, such that substitution of

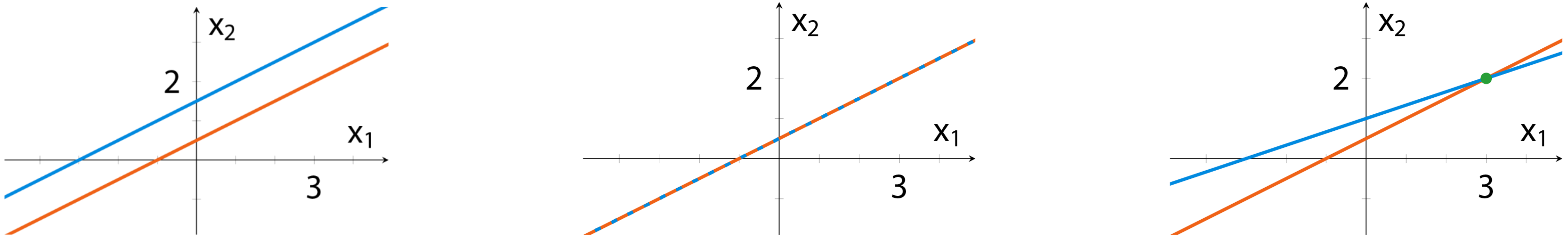
$$x_1 = c_1, x_2 = c_2, \ldots, x_n = c_n$$

into each of the equations yields a true identity.
The solution set or general solution of a system is the set of all solutions.

Example:
$(1, 2)$ is a solution of the system $\begin{cases} 3x_1 + x_2 & = & 5 \\ 5x_1 - 2x_2 & = & 1 \end{cases}$

Solutions of linear systems

Theorem:
A linear system has 0, 1 or infinitely many solutions.



Consistent vs. inconsistent system

Definition:

A linear system is called

- consistent   if it has at least one solution,
- inconsistent   if it has no solution at all

Solutions of linear systems

The following system has a unique solution (x_1, x_2, x_3)

$$\begin{cases} x_1 + 5x_2 + 3x_3 = 7 \\ \quad\; - 3x_2 + 3x_3 = -2\,. \\ \qquad\qquad\;\; 6x_3 = 2 \end{cases}$$

What is the value of x_1?

✓**A.** 1 **B.** $\frac{1}{3}$ **C.** -1 **D.** $-\frac{1}{3}$

Consistent system

How many solutions can a consistent system of linear equations have?

A. Zero or infinitely many

B. Zero

✓ **C.** One or infinitely many

D. One

E. Zero or one

Augmented matrix

The augmented matrix

A linear system is completely determined by the coefficients and the numbers on the right-hand side; the choice of the names of the variables for the unknowns is not essential.

$$\begin{cases} x_1 + 5x_2 + 3x_3 = 8 \\ 2x_1 + x_2 + 15x_3 = -11 \\ -x_1 + x_2 + 3x_3 = -2 \end{cases} \Rightarrow \left[\begin{array}{rrr|r} 1 & 5 & 3 & 8 \\ 2 & 1 & 15 & -11 \\ -1 & 1 & 3 & -2 \end{array}\right]$$

The part before the vertical bar, i.e. $A = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 1 & 15 \\ -1 & 1 & 3 \end{bmatrix}$

is called the **coefficient matrix** of the system.

The augmented matrix

$$\begin{cases} x_1 + 5x_2 + 3x_3 &= 8 \\ 2x_1 + x_2 + 15x_3 &= -11 \\ -x_1 + x_2 + 3x_3 &= -2 \end{cases} \quad \Rightarrow \quad \left[\begin{array}{rrr|r} 1 & 5 & 3 & 8 \\ 2 & 1 & 15 & -11 \\ -1 & 1 & 3 & -2 \end{array}\right]$$

Alternative notation for this system (used in book for Linear Circuits):

$$\begin{bmatrix} 1 & 5 & 3 \\ 2 & 1 & 15 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -11 \\ -2 \end{bmatrix}.$$

You will learn more about this notation, called the matrix-vector product, in the Q3 course Calculus and Linear Algebra.

Solutions of linear systems

For the system $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 3 & 9 \end{array}\right]$ the solution (x_1, x_2, x_3) is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

A. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

B. $\begin{bmatrix} 1 \\ 6 \\ 9 \end{bmatrix}$

✓**C.** $\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$

D. No solution exists

Solutions of linear systems

For the system $\begin{bmatrix} 1 & 5 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$ the solution (x_1, x_2, x_3) is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

A. $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ **B.** $\begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$ **C.** $\begin{bmatrix} -9 \\ 2 \\ 0 \end{bmatrix}$ ✓ **D.** No solution exists

Practice now

Exercise 1

Consider the systems (i) $\begin{cases} 2x_1 + x_2 - 4x_3 = 1, \\ -x_1 - 2x_2 + 4x_3 = 0, \\ 3x_1 + 2x_2 = 7, \end{cases}$ (ii) $\left[\begin{array}{rrr|r} 2 & 1 & -4 & 1 \\ -1 & -2 & 4 & 0 \\ 3 & 2 & 0 & 7 \end{array}\right]$ and

(iii) $\begin{bmatrix} 2 & -1 & 3 \\ 1 & -2 & 2 \\ -4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}$.

Two out of these three describe the same system of linear equations. How should the incorrect one be written instead?

Practice now - Answers

Exercise 1

Consider the systems (i) $\begin{cases} 2x_1 + x_2 - 4x_3 = 1, \\ -x_1 - 2x_2 + 4x_3 = 0, \\ 3x_1 + 2x_2 = 7, \end{cases}$ (ii) $\left[\begin{array}{ccc|c} 2 & 1 & -4 & 1 \\ -1 & -2 & 4 & 0 \\ 3 & 2 & 0 & 7 \end{array}\right]$ and

(iii) $\begin{bmatrix} 2 & -1 & 3 \\ 1 & -2 & 2 \\ -4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}.$

Two out of these three describe the same system of linear equations. How should the incorrect one be written instead?

(i) and (ii) are the same. (iii) should be: $\begin{bmatrix} 2 & 1 & -4 \\ -1 & -2 & 4 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}.$

Solving linear systems

Solutions of linear systems

Consider the systems (i) $\left[\begin{array}{ccc|c} 1 & 5 & 3 & 8 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -2 \end{array}\right]$, (ii) $\left[\begin{array}{ccc|c} 1 & 5 & 3 & 8 \\ 2 & 1 & 15 & -11 \\ -1 & 1 & 3 & -2 \end{array}\right]$ and (iii) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$.

For which of these systems can the solution be found with the lowest number of computation steps?

A. (i) **B.** (ii) ✓**C.** (iii)

Equivalent systems

Consider the linear system

$$\begin{cases} x_1 + 5x_2 + 3x_3 = 8 \\ 2x_1 + x_2 + 15x_3 = -11 \\ -x_1 + x_2 + 3x_3 = -2 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 5 & 3 & 8 \\ 2 & 1 & 15 & -11 \\ -1 & 1 & 3 & -2 \end{array}\right]$$

We will solve this system by bringing it to a simpler form without changing the solution set.

Two linear systems with the same solution set are called equivalent systems.

Equivalent systems

In fact, the systems $\left[\begin{array}{rrr|r} 1 & 5 & 3 & 8 \\ 2 & 1 & 15 & -11 \\ -1 & 1 & 3 & -2 \end{array}\right]$, $\left[\begin{array}{rrr|r} 1 & 5 & 3 & 8 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -2 \end{array}\right]$ and $\left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$ are all equivalent.

Main question: How do we get from the initial system to the (easier!) equivalent systems?

Elementary row operations

Elementary row operations:

a. One row is replaced by the sum of itself and a multiple of another row.

b. Two rows are interchanged.

c. One row is multiplied by a nonzero constant.

$$\left[\begin{array}{rrr|r} 1 & 5 & 3 & 8 \\ 2 & 1 & 15 & -11 \\ -1 & 1 & 3 & -2 \end{array}\right]\begin{array}{l} [R_1] \\ [R_2 - 2R_1] \\ [R_3] \end{array} \sim \left[\begin{array}{rrr|r} 1 & 5 & 3 & 8 \\ 0 & -9 & 9 & -27 \\ -1 & 1 & 3 & -2 \end{array}\right]\begin{array}{l} [R_1] \\ [R_3] \\ [R_2] \end{array} \sim \left[\begin{array}{rrr|r} 1 & 5 & 3 & 8 \\ -1 & 1 & 3 & -2 \\ 0 & -9 & 9 & -27 \end{array}\right]\begin{array}{l} [R_1] \\ [R_2] \\ [-\frac{1}{9}R_3] \end{array}$$

Performing elementary row operations does not change the solution set.

Elementary row operations

Which matrix is the result of the given row operation?

$$\left[\begin{array}{cc|c} 0 & -2 & 5 \\ 1 & 3 & -5 \\ 3 & -1 & 6 \end{array}\right] \begin{array}{c} [R_2] \\ [R_1] \\ [R_3] \end{array}$$

A. $\left[\begin{array}{cc|c} 0 & -2 & 5 \\ 1 & 3 & -5 \\ 3 & -1 & 6 \end{array}\right]$

✔ **C.** $\left[\begin{array}{cc|c} 1 & 3 & -5 \\ 0 & -2 & 5 \\ 3 & -1 & 6 \end{array}\right]$

B. $\left[\begin{array}{cc|c} 1 & 3 & -5 \\ 1 & 3 & -5 \\ 3 & -1 & 6 \end{array}\right]$

D. $\left[\begin{array}{cc|c} 0 & -6 & 15 \\ 1 & 3 & -5 \\ 3 & -1 & 6 \end{array}\right]$

Elementary row operations

Which matrix is the result of the given row operation?

$$\left[\begin{array}{cc|c} 1 & 3 & -5 \\ 0 & -2 & 5 \\ 3 & -1 & 6 \end{array}\right] \begin{array}{l} [R_1] \\ [R_2] \\ [R_3 - 3R_1] \end{array}$$

✓ **A.** $\left[\begin{array}{cc|c} 1 & 3 & -5 \\ 0 & -2 & 5 \\ 0 & -10 & 21 \end{array}\right]$

C. $\left[\begin{array}{cc|c} -8 & 6 & -23 \\ 0 & -2 & 5 \\ 3 & -1 & 6 \end{array}\right]$

B. $\left[\begin{array}{cc|c} 1 & 3 & -5 \\ 0 & -2 & 5 \\ 6 & 8 & 21 \end{array}\right]$

D. $\left[\begin{array}{cc|c} 1 & 3 & -5 \\ 0 & -2 & 5 \\ 3 & -1 & 21 \end{array}\right]$

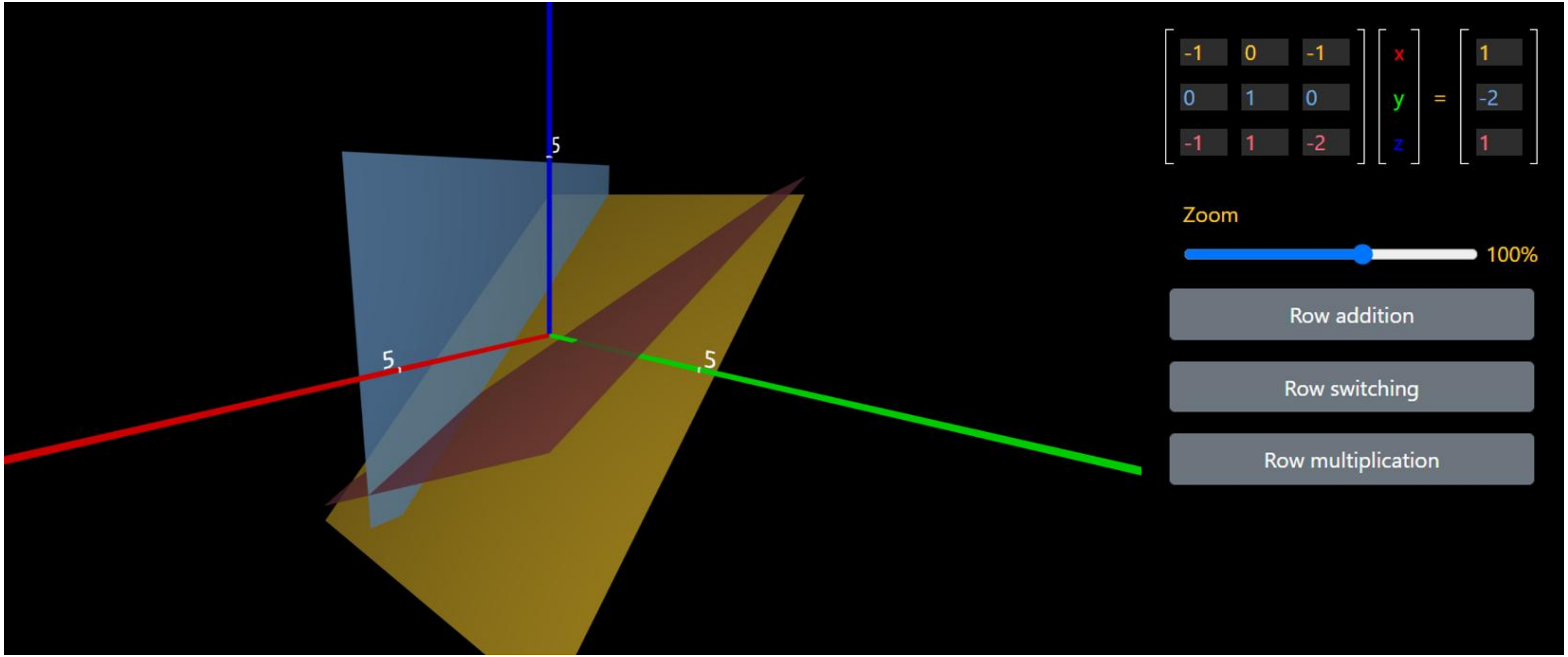
Elementary row operations

Definition:
Matrices that can be transformed into each other via row operations are called row equivalent. If two matrices A and B are row equivalent we denote this by $A \sim B$.

Note:
If two augmented matrices are row equivalent it means that the linear systems they represent are equivalent, i.e. have the same solution set.

Applet Row Reduction

Click on the image to use this applet!



(Reduced) Echelon forms

Echelon form

Definition:

A matrix is in echelon form if it has the following three properties:

a. All nonzero rows are above any row of all zeros.
b. Each leading entry (pivot) is in a column to the right of the leading entry in the previous row.
c. All entries below a leading entry are zero.

$$\left[\begin{array}{ccc|c} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

Reduced echelon form

Definition:

A matrix is in reduced echelon form if it has the following three properties:

a. It is in echelon form.
b. The pivot of each nonzero row is 1.
c. Each leading 1 is the only nonzero entry in its column.

$$\left[\begin{array}{ccc|c} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & * & 0 \\ 0 & 0 & 0 & 1 & 0 & * & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 & * & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Echelon form

Is the following matrix in echelon form?

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 4 \\ 0 & 5 & 7 & 8 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 0 \end{array}\right]$$

A. Yes

✓ **B.** No

Echelon form

Given the matrix below. The matrix is:

$$\left[\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & * & * & * & * \\ 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & 0 \end{array}\right]$$

A. In reduced echelon form

✓**B.** In echelon form, but not necessarily reduced

C. Not in echelon form

Row reduction: Examples of different cases

$$\left[\begin{array}{rrr|r} 1 & 5 & 3 & 8 \\ 2 & 1 & 15 & -11 \\ -1 & 1 & 3 & -2 \end{array}\right]$$

One solution

$$\left[\begin{array}{rrr|r} 1 & 5 & 3 & 8 \\ 2 & 1 & 15 & -11 \\ -1 & 1 & -9 & -2 \end{array}\right]$$

No solution

$$\left[\begin{array}{lrr|r} 0 & 1 & 4 & -3 \\ 2 & -1 & 4 & -3 \\ 4 & -1 & 12 & -9 \end{array}\right]$$

Infinitely many solutions

Solving a system of linear equations

1. Write down the augmented matrix corresponding to the system.
2. Row reduce the augmented matrix to echelon form (see next slide).
3. Determine whether or not the system is consistent.
4. In case of consistency, find the solution(s) by:
   a. backward substitution or;
   b. further row reduction of the augmented matrix to reduced echelon form.

Solving a system of linear equations

2. Row reduce the augmented matrix to echelon form.
    i. Start with the leftmost nonzero column. Select a nonzero entry in this column as a pivot. If necessary, interchange rows to move this entry to the pivot position at the top.
    ii. Use row replacements operations to create zeroes in all positions below the pivot.
    iii. Ignore the row with the pivot position and repeat the previous steps to the smaller matrix that remains. Repeat until there are no more nonzero rows to modify.

Row reduction algorithm

Note:
If a system is inconsistent then the row reduction algorithm will produce at least one row of the form

$$\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & c \end{array}\right]$$

with $c \neq 0$.

Remark:
The row reduction algorithm is also known as **Gaussian elimination**.

Practice now

Exercise 1

In a linear circuit, the voltages (v_1, v_2, v_3) at three certain locations in the system satisfy the following stystem of linear equations.

$$\begin{cases} v_1 + 3v_2 + 4v_3 = -5, \\ -v_1 + 2v_2 + 2v_3 = -3, \\ 2v_1 \qquad\quad + 2v_3 = -4. \end{cases}$$

Find the values of v_1, v_2, v_3.

Practice now - Answers

Exercise 1

In a linear circuit, the voltages (v_1, v_2, v_3) at three certain locations in the system satisfy the following stystem of linear equations.

$$\begin{cases} v_1 + 3v_2 + 4v_3 = -5, \\ -v_1 + 2v_2 + 2v_3 = -3, \\ 2v_1 \qquad\quad + 2v_3 = -4. \end{cases}$$

Find the values of v_1, v_2, v_3. $v_1 = 1\,V, v_2 = 2\,V, v_3 = -3\,V$

Wrap-up and next lecture

After practicing the topics of this lecture you are able to:

- Switch notation between a linear system, an augmented matrix and a matrix-vector product;
- Solve a linear system by row reducing it into echelon form;
- Determine whether a linear system has zero, one or infinitely many solutions.

Next lecture:

Limits and continuity - computations

Stewart §2.3 until "the squeeze theorem", §2.5 until "the Intermediate value theorem" and §2.6 until "Infinite limits at infinity"

See you next lecture!

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