

EE1C2 “Linear Circuits B”

Week 2.1

Francesco Fioranelli / Ioan E. Lager

The EE1C2 team...



Today

- Recapitulation of Q1
- New topics:
 - Sinusoidal voltages and currents
 - Phasors, phasor relations for R , L and C
 - Impedance, admittance
 - Analysis examples
 - Exam exercise example
- Summary of the day
- Next tasks

Recapitulation of Q1

- You became conversant in the full battery of circuit analysis methods
- You have a full dictionary of circuit elements: sources, resistances, capacitances, inductances and op amps
- Capacitances and inductances react to changes in circuits \Rightarrow you examined transients in first- and second-order circuits (transition from a steady-state to another steady-state)
- But what will happen if (smooth) change **is** the steady-state?

Sinusoidal voltages and currents

Sinusoids / sinusoidal functions

- Sinusoidal function

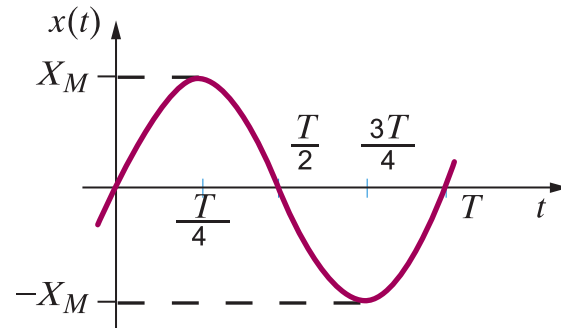
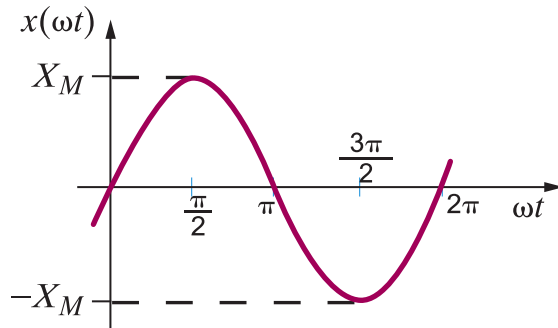
$$x(t) = X_M \sin \omega t$$

$$x[\omega(t + T)] = x(\omega t)$$

- X_M = the amplitude

- ω = the angular frequency $\longrightarrow \omega t$ = the argument

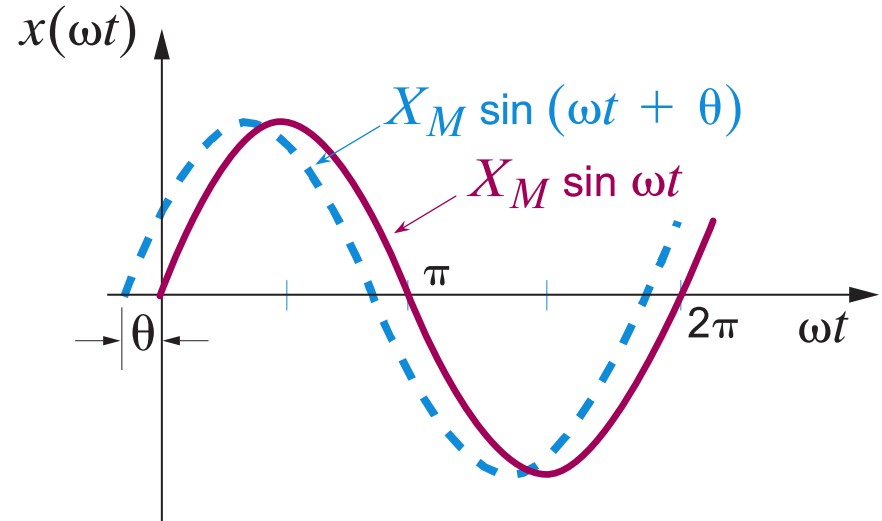
- The plot as a function of ωt or as a function of t



Sinusoids / sinusoidal functions

- Basic features: (angular) frequency $f = \frac{1}{T} \implies \omega = \frac{2\pi}{T} = 2\pi f$
- Phase (argument):

$$x(t) = X_M \sin(\underbrace{\omega t + \theta})$$



Sinusoids / sinusoidal functions

- Important:

$$x_1(t) = X_M \sin(\omega t + \theta) \quad x_2(t) = X_M \sin(\omega t + \phi)$$

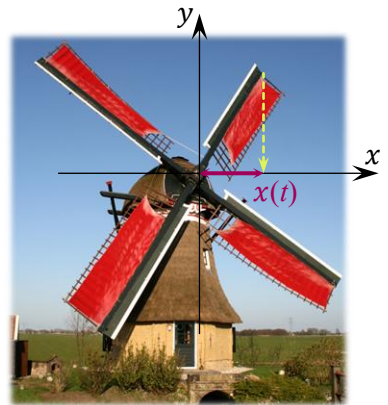
- Assume that $\theta > \phi$; we say that:

- x_1 leads x_2 by $\theta - \phi$ radians
- x_2 lags x_1 by $\phi - \theta$ radians

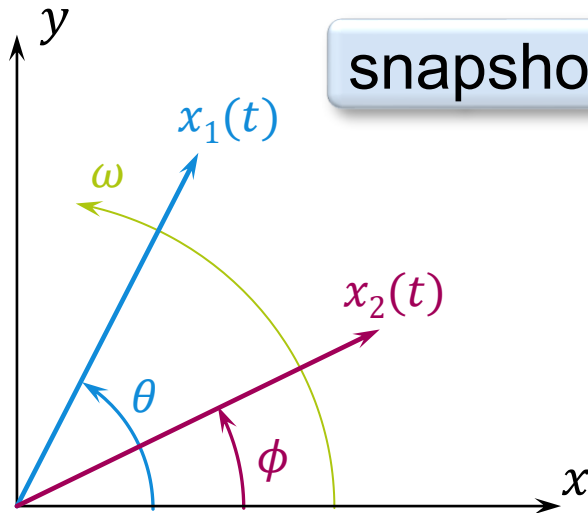
$$\sin(\omega t) = \cos(\omega t - \pi/2)$$

$$a \cos(\omega t) + b \sin(\omega t) = \frac{a}{\cos(\theta)} \cos(\omega t - \theta)$$

with $\theta = \arctan(b/a)$, ($a \neq 0$) ...more in the book



snapshot!

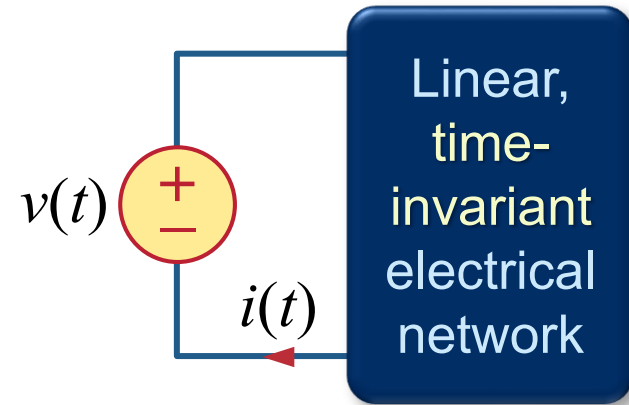


Sinusoidal and complex sources

- The steady-state voltages and currents in circuits fed by sinusoidal sources are themselves sinusoidal

Consequence of KVL and KCL

- If $v(t) = A \sin(\omega t + \theta)$
then the current must be of the form
 $i(t) = B \sin(\omega t + \phi)$



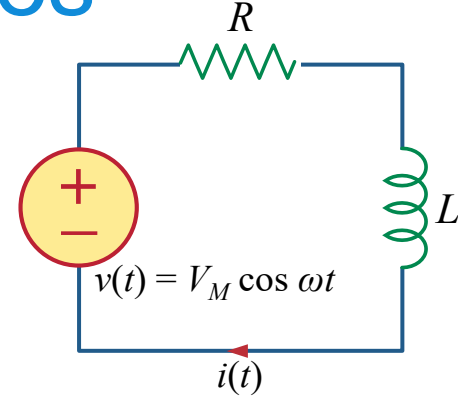
The basis for analysing steady-state AC circuits

Sinusoidal and complex sources

- Example: RL circuit:

differential equation: $L \frac{di(t)}{dt} + Ri(t) = V_M \cos \omega t$

$$i(t) = \frac{V_M}{\sqrt{R^2 + \omega^2 L^2}} \cos \left(\omega t - \tan^{-1} \frac{\omega L}{R} \right)$$



- Phase shift:

- if $L = 0 \Rightarrow i(t)$ is in phase with $v(t)$
- if $R = 0 \Rightarrow i(t)$ lags behind $v(t)$ by 90°
- if L and R are both non-zero $\Rightarrow i(t)$ lags behind $v(t)$ by a value between 0° and 90°

Sinusoidal and complex sources

- It is abundantly clear: even a simple RL circuit involves a lot of work
- This can be simpler!
via the relation between sinusoidal functions and complex functions
- Euler relation:

$$e^{j\omega t} = \cos \omega t + j \sin \omega t = \operatorname{Re}\{e^{j\omega t}\} + j \operatorname{Im}\{e^{j\omega t}\}$$



Leonhard Euler
15 April 1707 –
18 September 1783

Sinusoidal and complex sources

- Assume a **non-physical** voltage source $v(t) = V_M e^{j\omega t}$

$$v(t) = V_M \cos \omega t + jV_M \sin \omega t$$

- The current response can be written by making use of linearity and superposition

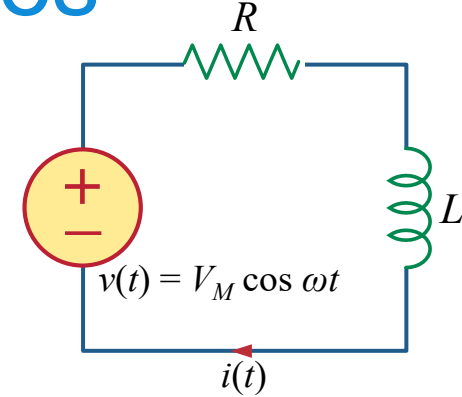
$$i(t) = I_M \cos(\omega t + \phi) + jI_M \sin(\omega t + \phi)$$

- And thus: $i(t) = I_M e^{j(\omega t + \phi)}$

Sinusoidal and complex sources

Is this really easier?

- Assume again $L \frac{di(t)}{dt} + Ri(t) = V_M \cos \omega t$
- We now apply $V_M e^{j\omega t}$ instead of $V_M \cos(\omega t)$
- The current response must be of the form



$$i(t) = I_M e^{j(\omega t + \phi)}$$

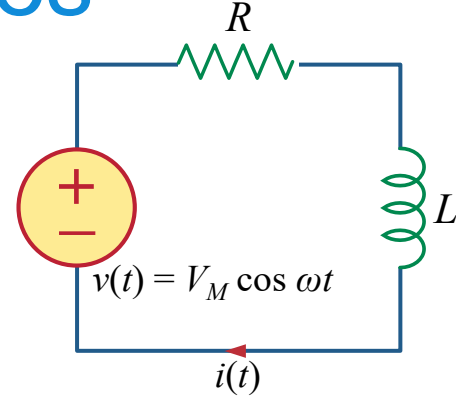
Sinusoidal and complex sources

- By filling in the quantities in the differential

equation $L \frac{di(t)}{dt} + Ri(t) = V_M \cos \omega t$

$I_M e^{j(\omega t + \phi)}$

$V_M e^{j\omega t}$



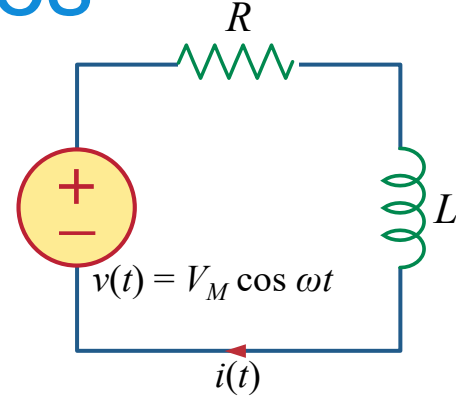
- It is found that:

$$j\omega L I_M e^{j(\omega t + \phi)} + R I_M e^{j(\omega t + \phi)} = V_M e^{j\omega t}$$

Sinusoidal and complex sources

- Divide by $e^{j\omega t} \longrightarrow j\omega LI_M e^{j\phi} + RI_M e^{j\phi} = V_M$

and do the algebra $\longrightarrow I_M e^{j\phi} = \frac{V_M}{R + j\omega L}$



- Express the complex result in polar coordinates:

$$I_M e^{j\phi} = \frac{V_M}{\sqrt{R^2 + \omega^2 L^2}} e^{j[-\tan^{-1}(\omega L / R)]}$$

Sinusoidal and complex sources

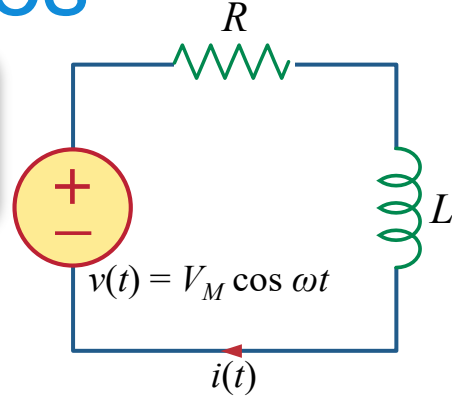
- Thus $I_M = \frac{V_M}{\sqrt{R^2 + \omega^2 L^2}}$ & $\phi = -\tan^{-1} \frac{\omega L}{R}$

- But $I_M e^{j\phi} = \frac{V_M}{\sqrt{R^2 + \omega^2 L^2}} e^{j[-\tan^{-1}(\omega L/R)]}$

is complex \longrightarrow non-physical!

- We put back the $e^{j\omega t}$ part and take the real part of the result

$$i(t) = I_M \cos(\omega t + \phi) = \frac{V_M}{\sqrt{R^2 + \omega^2 L^2}} \cos\left(\omega t - \tan^{-1} \frac{\omega L}{R}\right)$$



Phasors

Phasors

- We observed that $e^{j\omega t}$ occurs overall without itself changing
 - we can ignore this term and only keep in mind the frequency
 - we then carry on in terms of the magnitude and phase, only
- We write $v(t) = V_M \cos(\omega t + \theta) = \text{Re}\left[V_M e^{j(\omega t + \theta)}\right]$
as $v(t) = \text{Re}\left(V_M \angle \theta e^{j\omega t}\right)$ and ignore (for now) $e^{j\omega t}$

The direct phasor transform


$$V_M \angle \theta$$

Phasors

- Phasors are usually typeset as boldface letters $\mathbf{V} = V_M \angle \theta$
- The differential equation in the example

$$L \frac{di(t)}{dt} + Ri(t) = V_M \cos \omega t \xrightarrow{\text{formal substitution}} L \frac{d}{dt} (\mathbf{I} e^{j\omega t}) + R \mathbf{I} e^{j\omega t} = \mathbf{V} e^{j\omega t}$$

becomes after dividing by $e^{j\omega t}$: $j\omega L \mathbf{I} + R \mathbf{I} = \mathbf{V}$

- Rules for operating with phasors:
 - derivative: $d/dt \longrightarrow (j\omega)$
 - integral: $\int \dots dt \longrightarrow (1/j\omega)$

Phasors

- Applying these rules entails that

$$\mathbf{I} = \frac{\mathbf{V}}{j\omega L + R} = I_M \angle \phi = \frac{V_M}{\sqrt{R^2 + \omega^2 L^2}} \angle -\tan^{-1} \frac{\omega L}{R}$$

- The real current can then be expressed as

$$i(t) = \frac{V_M}{\sqrt{R^2 + \omega^2 L^2}} \cos\left(\omega t - \tan^{-1} \frac{\omega L}{R}\right)$$

The inverse phasor transform

Phasors

- Summarising:
 - at a first glance, this method seems more intricate
 - but it allows replacing differential equations by algebraic equations that are clearly easier to handle
- Algorithm:
 - 1) Write the differential equations and transform them into algebraic ones by making use of phasors and the derivative/integral rules
 - 2) Solve the algebraic equations
 - 3) Transform the derived phasors back to the time-domain

Phasors

- Summarising
 - at a first glance
 - but it allows r that are clear
- Algorithm:
 - 1) Write the **di ones** by ma
 - 2) Solve the a
 - 3) Transform t

COMPLEX QUANTITIES AND THEIR USE IN ELECTRICAL ENGINEERING.

BY CHAS. PROTEUS STEINMETZ.

I.—INTRODUCTION.

In the following, I shall outline a method of calculating **alternate current phenomena**, which, I believe, differs from former methods essentially in so far, as it allows us to represent the **alternate current**, the sine-function of time, by a *constant* numerical quantity, and thereby eliminates the independent variable “time” altogether from the calculation of **alternate current phenomena**.

Herefrom results a considerable simplification of methods.

C.P. Steinmetz, “Complex quantities and their use in electrical engineering,” 1893

Phasors

- Summarising:

A (not so) small detail concerning the phasor electrical quantities $\mathbf{V} = V_M \angle \theta_v$ and $\mathbf{I} = I_M \angle \theta_i$

- V_M, I_M = magnitudes (amenable to measurement)
- θ_v, θ_i = phases (actually, time delays)
- the combination of the two = a complex number

Assigning a measure unit to a phasor (or any complex number) makes no physical sense! (still, it is often done...)

Phasor relations

- We now want to deal with circuits by using phasors
- We also want to establish in this way relations between R , L and C

Phasor relations

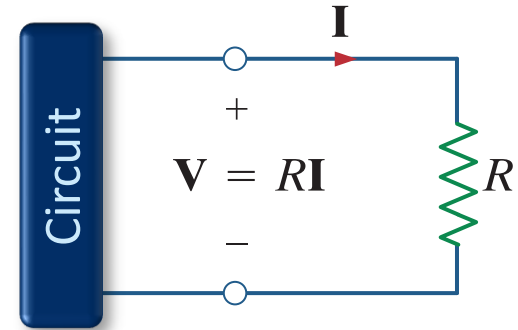
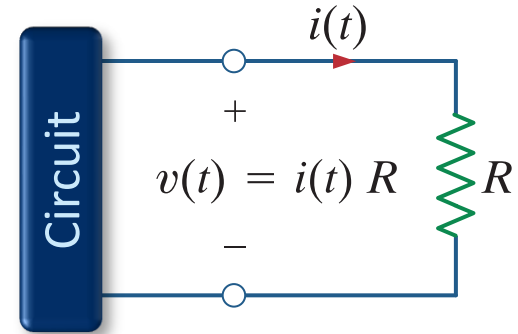
- Resistances: $v(t) = Ri(t)$

$$V_M e^{j(\omega t + \theta_v)} = RI_M e^{j(\omega t + \theta_i)}$$

$$V_M e^{j\theta_v} = RI_M e^{j\theta_i}$$

- In phasor form: $\mathbf{V} = R\mathbf{I}$

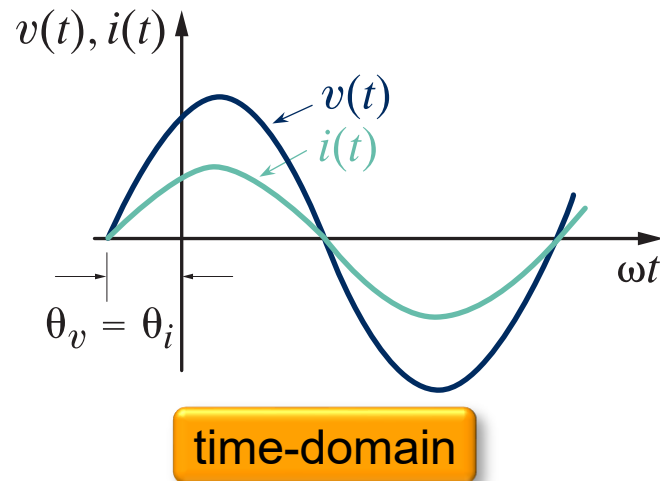
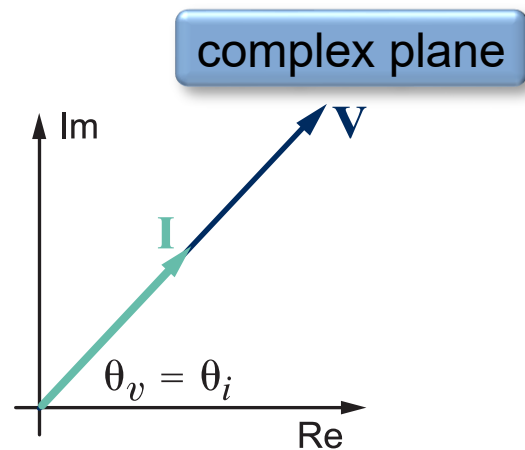
$$\mathbf{V} = V_M e^{j\theta_v} = V_M \angle \theta_v \quad \mathbf{I} = I_M e^{j\theta_i} = I_M \angle \theta_i$$



Phasor relations

$$\mathbf{V} = R\mathbf{I} \quad (\mathbf{V} = V_M e^{j\theta_v} = V_M \angle \theta_v, \quad \mathbf{I} = I_M e^{j\theta_i} = I_M \angle \theta_i)$$

- The phases θ_v and θ_i are equal \longrightarrow
 V and I are “in phase”
- This can be cast into a phasor diagram
 - amplitude ratios
 - phase shifts
 - which phasor leads or lags?



Phasor relations

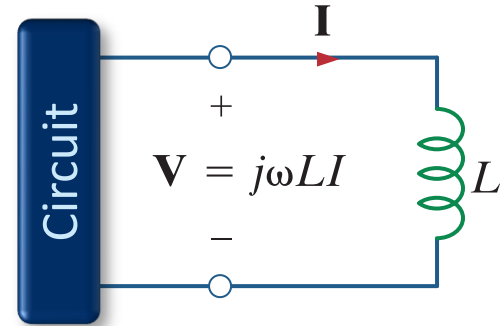
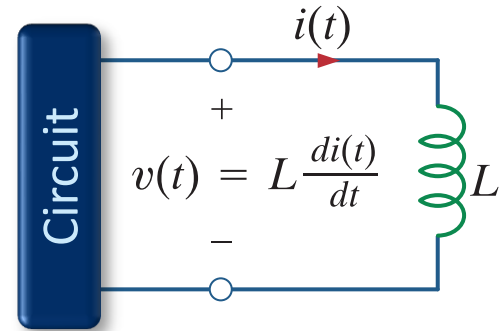
- Inductances: $v(t) = L \frac{di(t)}{dt}$
- Again, the same steps:

$$V_M e^{j(\omega t + \theta_v)} = L \frac{d}{dt} I_M e^{j(\omega t + \theta_i)}$$



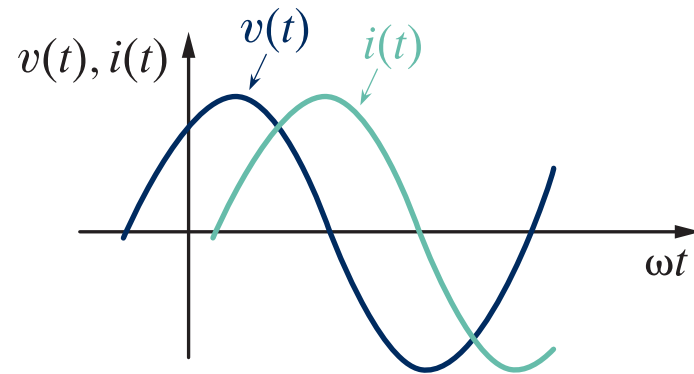
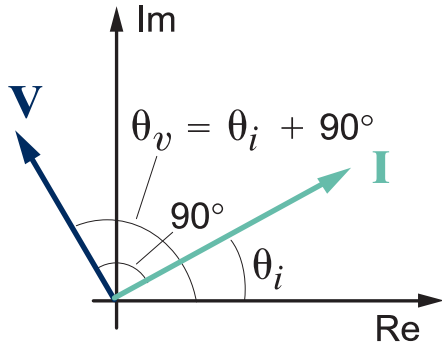
$$V_M e^{j\theta_v} = j\omega L I_M e^{j\theta_i}$$

$$\mathbf{V} = j\omega L \mathbf{I}$$



Phasor relations

- For j it holds that $j = 1e^{j90^\circ} = 1\angle 90^\circ$
- $V_M e^{j\theta_v} = j\omega L I_M e^{j\theta_i}$ then becomes $V_M e^{j\theta_v} = \omega L I_M e^{j(\theta_i + 90^\circ)}$
- The voltage thus **leads** the current by 90°

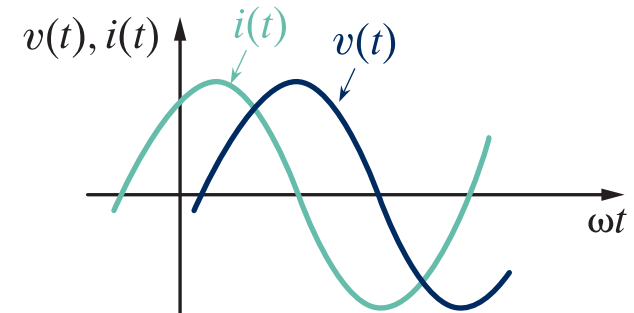
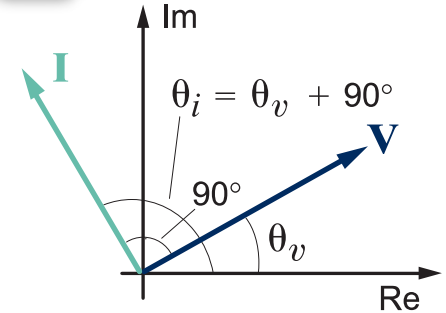
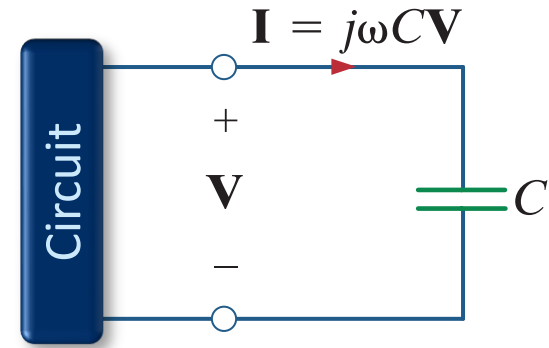


Phasor relations

- Capacities: similar derivation

$$I_M e^{j\theta_i} = j\omega C V_M e^{j\theta_v} \longrightarrow \boxed{\mathbf{I} = j\omega C \mathbf{V}}$$

- Here, it holds that $I_M e^{j\theta_i} = \omega C V_M e^{j(\theta_v + 90^\circ)}$
- The current thus leads the voltage by 90°



Phasors \longleftrightarrow phasor transform

- Sinusoidal feeding \longrightarrow sinusoidal response
- The term $e^{j\omega t}$ occurs in all equations without changing
 - we ignore this term and only keep in mind the frequency
 - calculations are done for the magnitude and phase, only
- We then write:

$$v(t) = V_M \cos(\omega t + \theta) = \operatorname{Re} \left[V_M e^{j(\omega t + \theta)} \right] = \operatorname{Re} \left(\underbrace{V_M \angle \theta}_{\text{phasor}} \cdot e^{j\omega t} \right)$$

Phasors \longleftrightarrow phasor transform

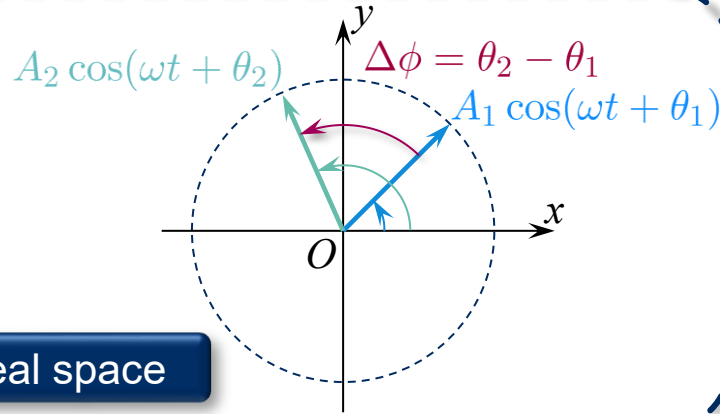
- Sinusoidal feeding \longrightarrow sinusoidal response
- The term $e^{j\omega t}$ occurs in all equations without changing

The phasor transform applies to time-depending quantities, only

- we then write.

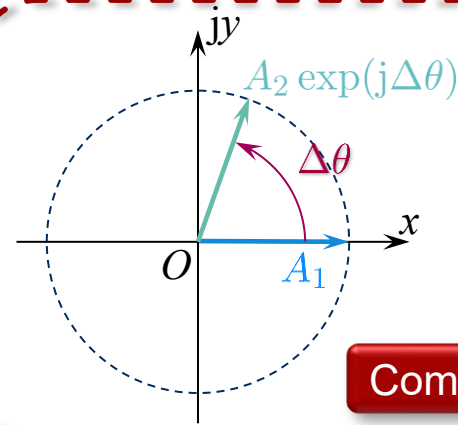
$$v(t) = V_M \cos(\omega t + \theta) = \text{Re} \left[V_M e^{j(\omega t + \theta)} \right] = \text{Re} \left(\underbrace{V_M \angle \theta}_{\text{phasor}} \cdot e^{j\omega t} \right)$$

Phasors ↔ phasor transform



Real space

No complex value
(no j)



Complex space

No ωt

Impedances

Impedances and admittances

- Impedance = an analogous of the time-domain resistance that is valid in the frequency-domain

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}}$$

- Since \mathbf{V} and \mathbf{I} are both complex $\Rightarrow \mathbf{Z}$ is also complex

$$\mathbf{Z} = \frac{V_M \angle \theta_v}{I_M \angle \theta_i} = \frac{V_M}{I_M} \angle \theta_v - \theta_i = Z \angle \theta_z$$

- The measure unit of $Z = |\mathbf{Z}|$ is ohm

Impedances and admittances

- In Cartesian coordinates, the impedance reads

$$\mathbf{Z}(\omega) = R(\omega) + jX(\omega)$$

with $R(\omega)$ being the **resistive component** (real part) and $X(\omega)$ the **reactive component** (imaginary part)

- $R(\omega)$ and $X(\omega)$ are real functions of $\omega \Rightarrow \mathbf{Z}$ is also a function of ω
- **\mathbf{Z} is not a phasor** \Rightarrow the phasor transform applies to time-dependent quantities, whereas \mathbf{Z} is a ratio (a constant)

Impedances and admittances

- Impedance parameters:

$$Z \angle \theta_z = R + jX \longrightarrow Z = \sqrt{R^2 + X^2} \quad \& \quad \theta_z = \tan^{-1} \frac{X}{R}$$

- Impedances (AC) \longleftrightarrow Resistances (DC) \longrightarrow
for series and parallel impedances it holds that

$$\mathbf{Z}_s = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 + \mathbf{Z}_4 + \dots + \mathbf{Z}_n$$

$$\frac{1}{\mathbf{Z}_p} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3} + \frac{1}{\mathbf{Z}_4} + \dots + \frac{1}{\mathbf{Z}_n}$$

Impedances and admittances

By accounting for phasor relations, resistances, inductances and capacities entail the following impedances

Element	Phasor Equation	Impedance
R	$\mathbf{V} = R\mathbf{I}$	$\mathbf{Z} = R$
L	$\mathbf{V} = j\omega L\mathbf{I}$	$\mathbf{Z} = j\omega L$
C	$\mathbf{V} = \frac{1}{j\omega C}\mathbf{I}$	$\mathbf{Z} = \frac{1}{j\omega C}$

Impedances and admittances

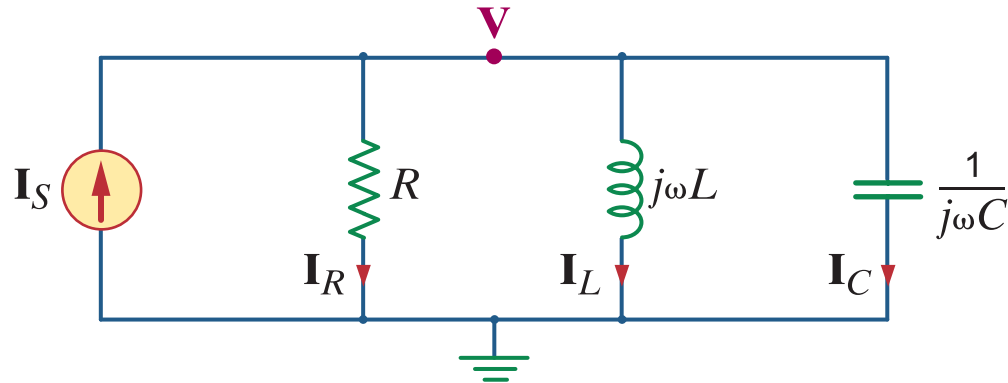
- The resistance's inverse is the conductance 
the impedance's inverse is **the admittance**

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}}$$

- the measure unit of $Y = |\mathbf{Y}|$ is siemens
- it is split-up as $\mathbf{Y} = Y\angle\theta_y = G + jB$ in which $G(\omega)$ is the conductance and $B(\omega)$ is the susceptance
- The same rules for series/parallel connections hold

Phasor diagrams

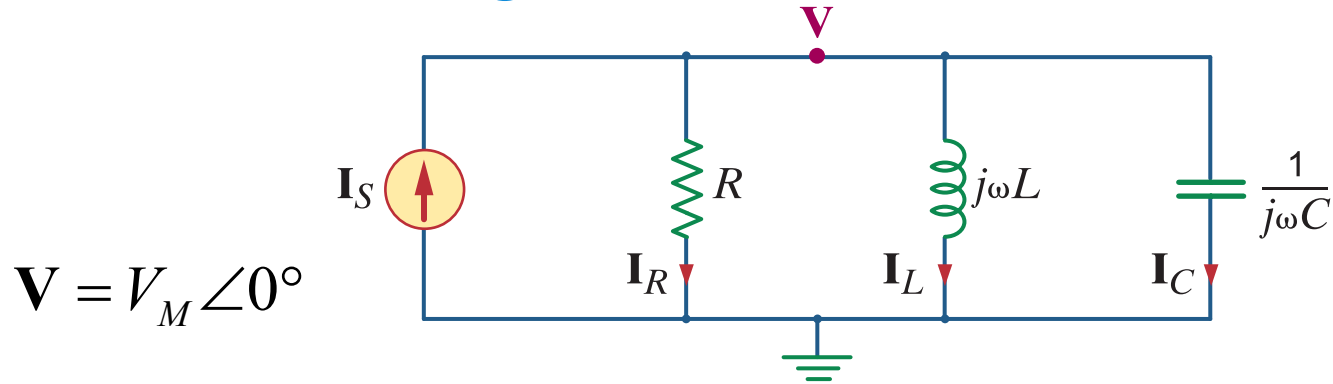
- Impedances and admittances are frequency dependent \longrightarrow so are the voltage \longleftrightarrow current relations in the network



- For the upper node it holds via KCL:

$$I_S = I_R + I_L + I_C = \frac{V}{Z_R} + \frac{V}{Z_L} + \frac{V}{Z_C} = \frac{V}{R} + \frac{V}{j\omega L} + \frac{V}{1/j\omega C}$$

Phasor diagrams



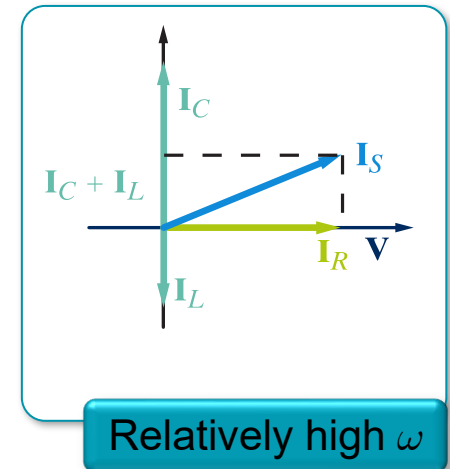
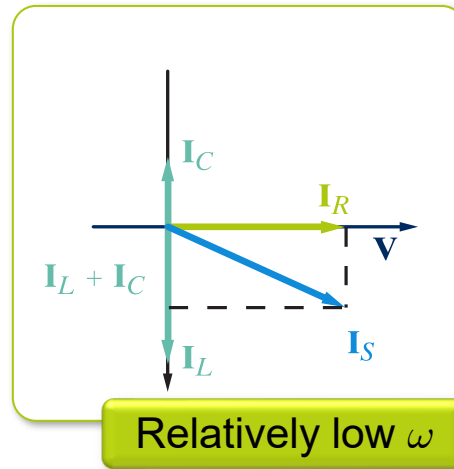
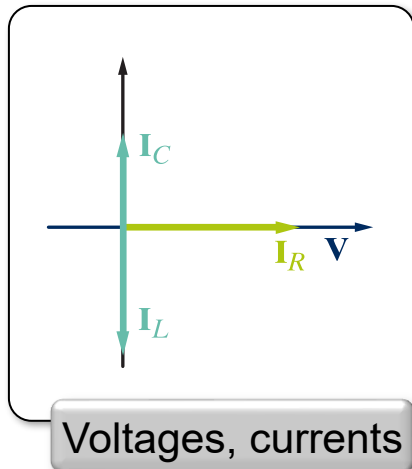
- We take: $V = V_M \angle 0^\circ$ (origin of the phase)

- It then follows that: $I_S = \frac{V_M \angle 0^\circ}{R} + \frac{V_M \angle -90^\circ}{\omega L} + V_M \omega C \angle 90^\circ$

Phasor diagrams

$$\mathbf{I}_S = \frac{V_M \angle 0^\circ}{R} + \frac{V_M \angle -90^\circ}{\omega L} + V_M \omega C \angle 90^\circ$$

- This can be represented graphically as **phasor diagrams**

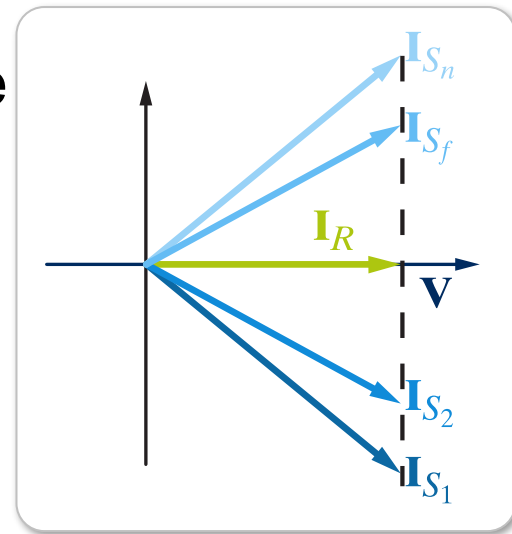


Phasor diagrams

$$\mathbf{I}_S = \frac{V_M \angle 0^\circ}{R} + \frac{V_M \angle -90^\circ}{\omega L} + V_M \omega C \angle 90^\circ$$

- As frequency increases, \mathbf{I}_S moves from bottom to the top
- When $\mathbf{I}_C = \mathbf{I}_L$, then \mathbf{I}_S and \mathbf{V} are in phase
 - this occurs at $\omega = \frac{1}{\sqrt{LC}}$
 - this can also be derived from KCL

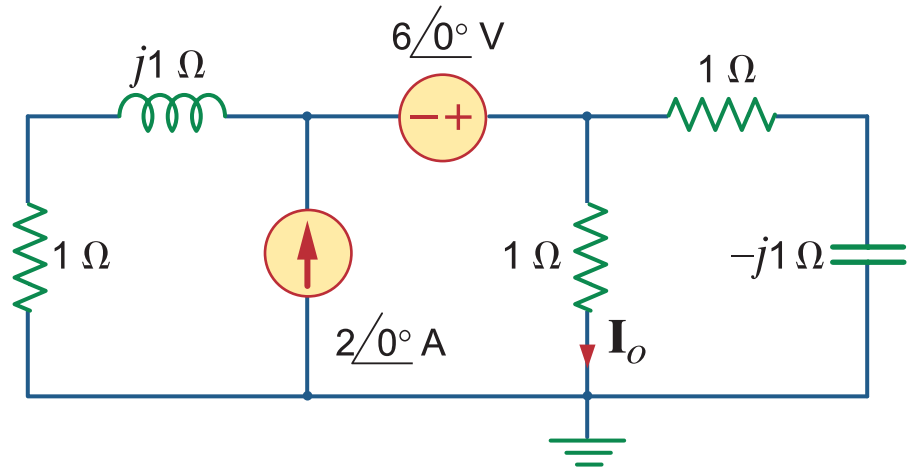
$$\mathbf{I}_S = \left[\frac{1}{R} + j \left(\omega C - \frac{1}{\omega L} \right) \right] \mathbf{V}$$



Examples

Example 1

Consider the circuit at the right. Determine I_o .



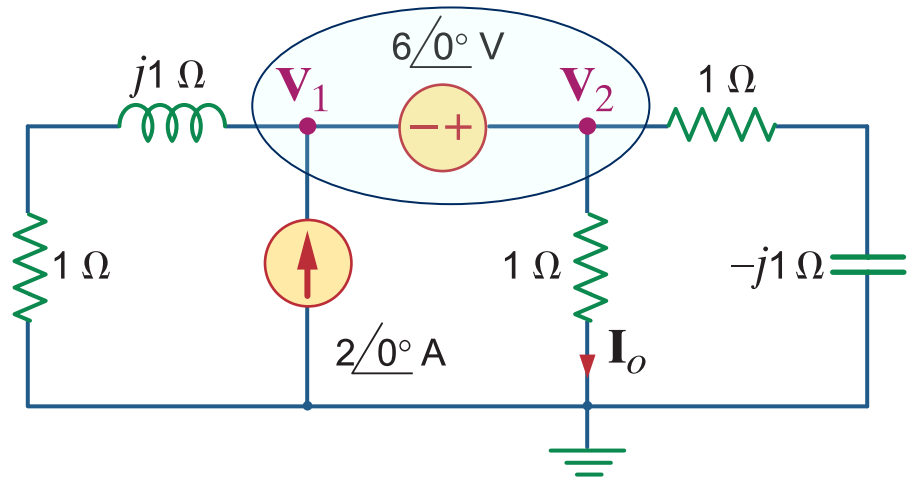
- Preliminaries:
 - sources transformed to the phasor domain (extract ω , the two sources are in phase)
 - calculate the values of the impedances: $R \rightarrow R$, $L \rightarrow j\omega L$, $C \rightarrow 1/j\omega C = -j/\omega C$
 - these operations are part of the exercise!

Example 1

Consider the circuit at the right. Determine I_o .

- One can opt for various solution techniques \longrightarrow
choice: nodal analysis

- Super-node: $\frac{V_1}{1+j} - 2\angle 0^\circ + \frac{V_2}{1} + \frac{V_2}{1-j} = 0$ & $V_1 + 6\angle 0^\circ = V_2$
- Substitution: $\frac{V_2 - 6\angle 0^\circ}{1+j} - 2\angle 0^\circ + V_2 + \frac{V_2}{1-j} = 0$



Example 1

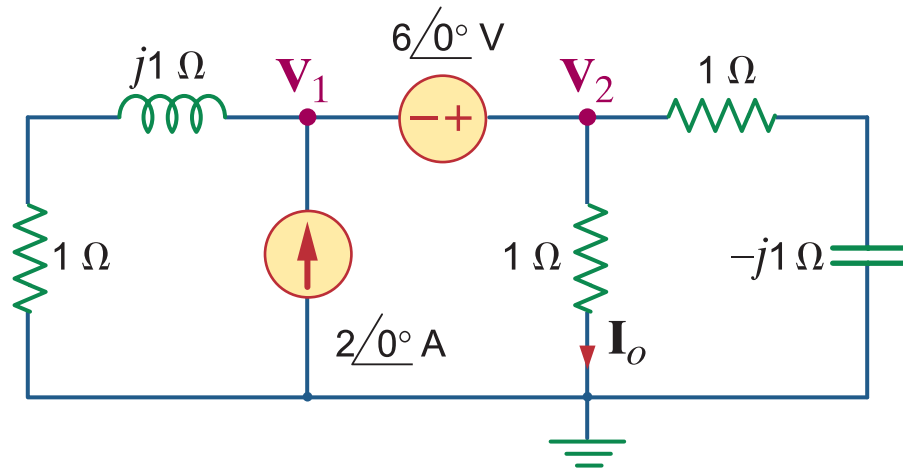
- Algebraic manipulation:

$$\frac{V_2 - 6\angle 0^\circ}{1+j} - 2\angle 0^\circ + V_2 + \frac{V_2}{1-j} = 0$$

$$V_2 \left[\underbrace{\frac{1}{1+j}}_{=2} + 1 + \underbrace{\frac{1}{1-j}}_{=2} \right] = \frac{6+2+2j}{1+j}$$

– this entails: $V_2 = \left(\frac{4+j}{1+j} \right)$

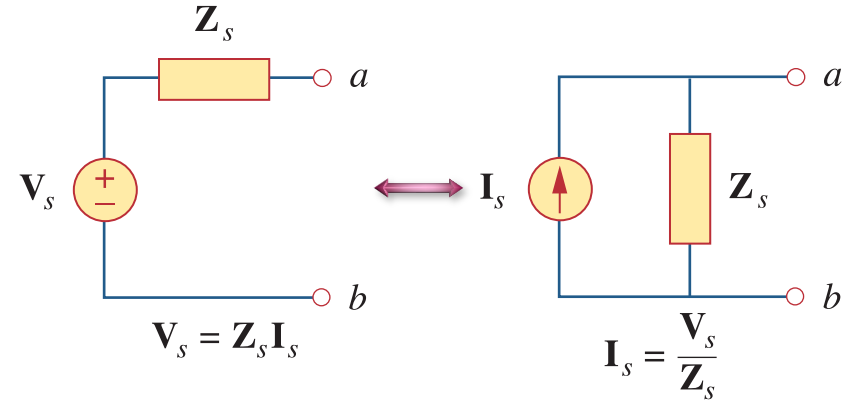
- Thus: $I_o = \left(\frac{4+j}{1+j} \right) = \frac{\sqrt{4^2+1^2}}{\sqrt{1^2+1^2}} \angle \left(\tan^{-1} \frac{1}{4} - \tan^{-1} \frac{1}{1} \right) = \sqrt{8.5} \angle -31^\circ$



Example 2

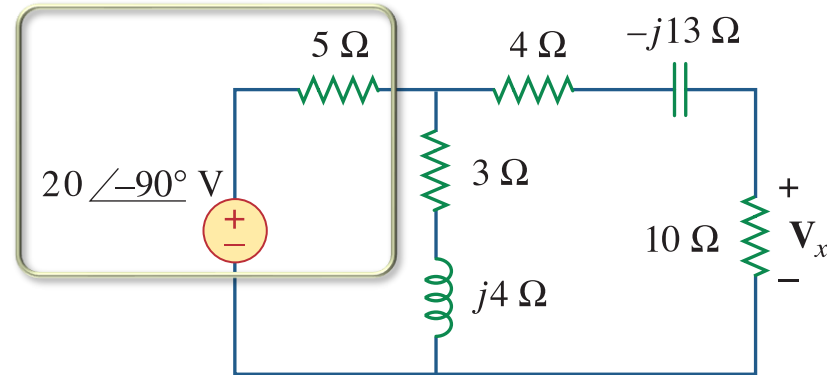
- Source transformation

$$\mathbf{V}_s = \mathbf{Z}_s \mathbf{I}_s \longleftrightarrow \mathbf{I}_s = \frac{\mathbf{V}_s}{\mathbf{Z}_s}$$



- Consider the circuit at the right. Determine \mathbf{V}_x .
- Source transformation: $\mathbf{V}_s \rightarrow \mathbf{I}_s$

$$\mathbf{I}_s = \frac{20 \angle -90^\circ}{5} = 4 \angle -90^\circ = -j4 \text{ A}$$



Example 2

- Parallel impedances:

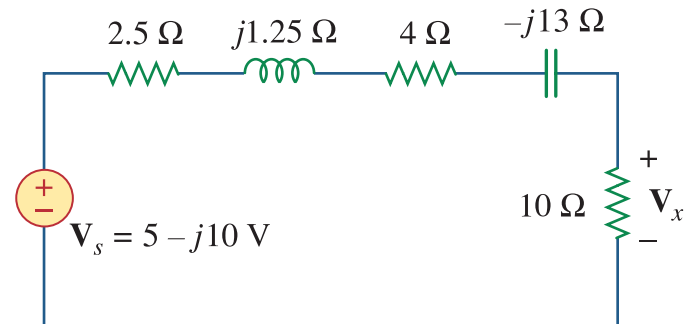
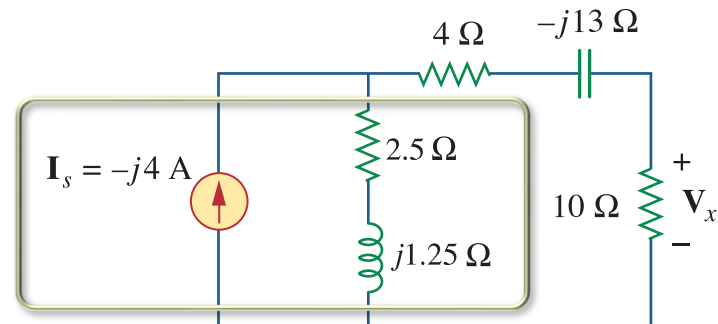
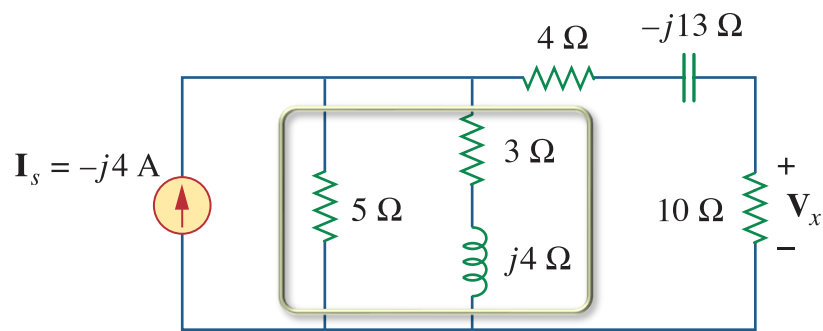
$$\mathbf{Z}_1 = \frac{5(3 + j4)}{8 + j4} = 2.5 + j1.25 \Omega$$

- Back to Thévenin: $\mathbf{V}_s \rightarrow \mathbf{I}_s$

$$\mathbf{V}_s = \mathbf{I}_s \mathbf{Z}_1 = -j4(2.5 + j1.25) = 5 - j10 \text{ V}$$

- Voltage division:

$$\mathbf{V}_x = \frac{10}{10 + 2.5 + j1.25 + 4 - j13} (5 - j10) \text{ V}$$



Example 2

- Algebraic manipulation:

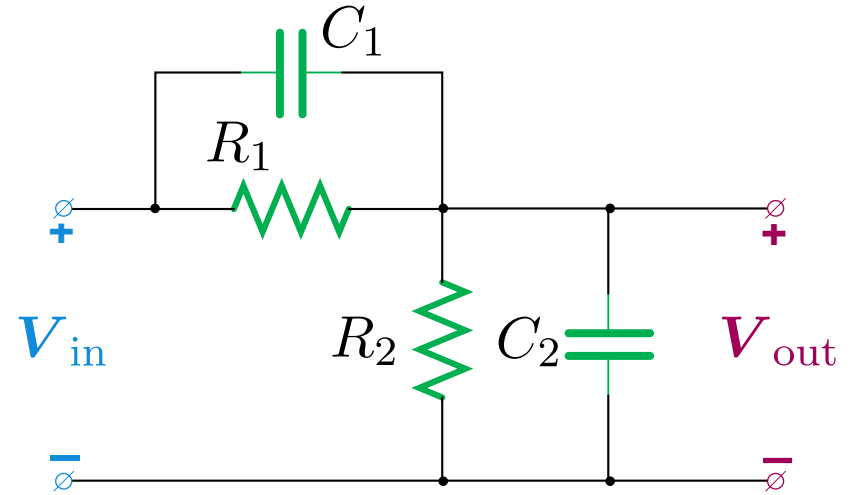
$$\begin{aligned}\mathbf{V}_x &= \frac{10}{10 + 2.5 + j1.25 + 4 - j13} (5 - j10) = \frac{50 - j100}{16.5 - j11.75} \\ &= \frac{\sqrt{50^2 + 100^2}}{\sqrt{16.5^2 + 11.75^2}} \left(\tan^{-1} \frac{-100}{50} - \tan^{-1} \frac{-11.75}{16.5} \right)\end{aligned}$$

$$= 5.52 \angle -28^\circ \text{ (V)}$$

Exam exercise example

Exam(ple)

- Consider the circuit in the figure at the right:



- Determine $\frac{V_{out}}{V_{in}}$.
- For which condition is $\frac{V_{out}}{V_{in}}$ frequency independent?

Exam(ple)

a) Determine $\frac{V_{\text{out}}}{V_{\text{in}}}$.

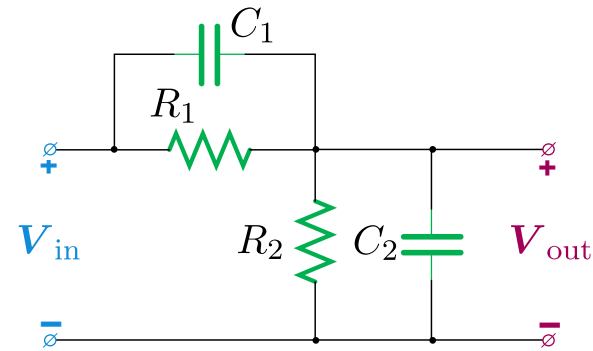
- Calculate the partial impedances:

$$- Z_1 = (R_1 \parallel C_1) = \frac{R_1 \frac{1}{j\omega C_1}}{R_1 + \frac{1}{j\omega C_1}} = \frac{R_1}{1 + j\omega R_1 C_1}$$

$$- Z_2 = (R_2 \parallel C_2) = \frac{R_2}{1 + j\omega R_2 C_2}$$

- Determine the transfer:

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{Z_2}{Z_1 + Z_2} = \frac{R_2(1 + j\omega R_1 C_1)}{R_1(1 + j\omega R_2 C_2) + R_2(1 + j\omega R_1 C_1)}$$



Exam(ple)

b) For which condition is $\frac{V_{\text{out}}}{V_{\text{in}}}$ frequency independent?

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{R_2(1 + j\omega R_1 C_1)}{R_1(1 + j\omega R_2 C_2) + R_2(1 + j\omega R_1 C_1)}$$

- For obtaining frequency independence, the terms containing ω must vanish from the expression
- By taking $R_1 C_1 = R_2 C_2 \implies \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{R_2}{R_1 + R_2}$ which is indeed frequency independent

Summary of the day

- Sinusoidal voltages and currents
- Phasors, phasor relations for R , L and C
- Impedance, admittance, phasor diagrams
- Analysis examples

Next tasks

- Please do the SGH1
- Seminars of Tuesday and Friday
- Next week: transfer functions

Thank you!