EE1C2 “Linear Circuits B”

Week 2.4

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Faculty of Electrical Engineering, Mathematics and Computer Science

Today

- Recap: passive and active filter networks
- Week 2.4:
  - Instantaneous power
  - Average power
  - Maximum average power
  - RMS value
- Summary and Next Week

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Recap of Week 2.3

- Filter Networks
  - Low-pass, high-pass, band-pass and band-stop (notch) filters
  - Resonant circuits, cutoff frequency, bandwidth, quality factor
  - Passive vs active filters

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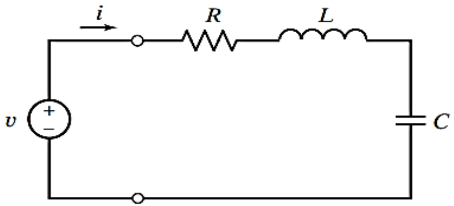
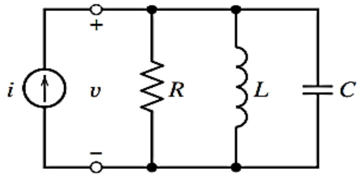
Ideal Filters

Type of Filter	$H(0)$	$H(\infty)$	$H(\omega_c)$ or $H(\omega_0)$
Lowpass	1	0	$1/\sqrt{2}$
Highpass	0	1	$1/\sqrt{2}$
Bandpass	0	0	1
Bandstop	1	1	0

ω_c is the cutoff frequency for lowpass and highpass filters; ω_0 is the center frequency for bandpass and bandstop filters.

Recall from the last lecture the analysis to the limit of ω to identify the type of filter!

Resonant Circuits

	SERIES RESONANT CIRCUIT	PARALLEL RESONANT CIRCUIT
Circuit		
Network function	$\mathbf{Y} = \dfrac{k}{1 + jQ\left(\dfrac{\omega}{\omega_0} - \dfrac{\omega_0}{\omega}\right)}$	$\mathbf{Z} = \dfrac{k}{1 + jQ\left(\dfrac{\omega}{\omega_0} - \dfrac{\omega_0}{\omega}\right)}$
Resonant frequency	$\omega_0 = \dfrac{1}{\sqrt{LC}}$	$\omega_0 = \dfrac{1}{\sqrt{LC}}$
Maximum magnitude	$k = \dfrac{1}{R}$	$k = R$
Quality factor	$Q = \dfrac{1}{R}\sqrt{\dfrac{L}{C}}$	$Q = R\sqrt{\dfrac{C}{L}}$
Bandwidth	$BW = \dfrac{R}{L}$	$BW = \dfrac{1}{RC}$

Recall from last lecture the ***definition of resonance****, which is valid* *<u>also for circuits where the components are not all in series or parallel</u>.*

-Instantaneous and Average Power
-Maximum Power Transfer

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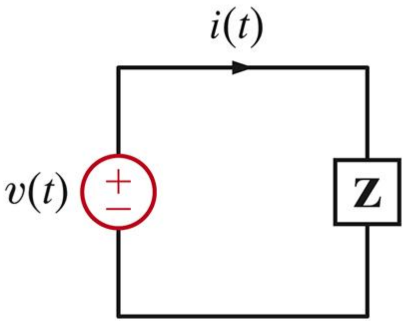
Instantaneous Power

With sinusoidal voltages and currents…

- We make use of the passive sign convention for calculating the **instantaneous** power which is by definition $p(t) = v(t)i(t)$
- If $v(t) = V_M \cos(\omega t + \theta_v)$ $i(t) = I_M \cos(\omega t + \theta_i)$

  Then the power can be rewritten as

$$p(t) = v(t)i(t) = V_M I_M \cos(\omega t + \theta_v)\cos(\omega t + \theta_i)$$



Here a trigonometric identity has been used

$$p(t) = \frac{V_M I_M}{2}\left[\cos(\theta_v - \theta_i) + \cos(2\omega t + \theta_v + \theta_i)\right]$$

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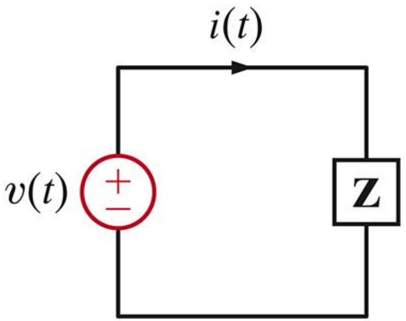
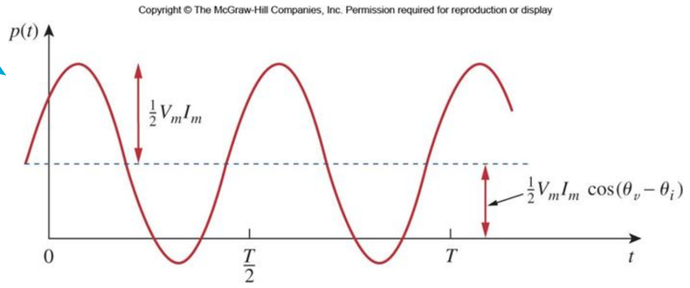
Instantaneous Power

Examining the cosine terms:

$$p(t)=\frac{V_M I_M}{2}\left[\cos(\theta_v-\theta_i)+\cos(2\omega t+\theta_v+\theta_i)\right]$$



- the first term is constant in time and depends on the phase difference between voltage and current
- the second term is time dependant, and fluctuates between *positive* values (the circuit absorbs power) and *negative* values (the circuit transfers power to the source…)
  - But…where is such power in the circuit coming from, where is it stored?



Average Power

- As the power changes with time, we can compute an average.
- The **average** power is calculated by integrating the instantaneous power over a full period *T* and dividing by *T*

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} p(t)dt$$

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} V_M I_M \cos(\omega t+\theta_v)\cos(\omega t+\theta_i)\,dt$$

Where t_0 is arbitrary, $T = 2\pi/\omega$, and *P* is measured in Watt

Average power is sometimes called **real power**, because it describes the power that is transformed from electric to nonelectric energy (e.g., heat).

Average Power

- The integral of the instantaneous power is manageable (right? If not ask…)

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} \frac{V_M I_M}{2}\left[\cos(\theta_v - \theta_i) + \cos(2\omega t + \theta_v + \theta_i)\right]dt$$

This becomes:

$$P = \frac{1}{2}V_M I_M \cos(\theta_v - \theta_i)$$

What happens to the average power if the circuit is made of only resistors, and of only inductors or capacitors?

- For a purely resistive network:

$$P = \frac{1}{2}V_m I_m = \frac{1}{2}I_m^2 R = \frac{1}{2}|I|^2 R$$

- For a purely reactive network:

$$P = \frac{1}{2}V_M I_M \cos(90°) = 0$$

Phasor notation, hence the absolute value

Purely Inductive and Capacitive Networks

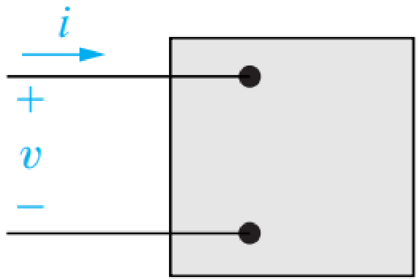
- For a purely inductive network: $\theta_v - \theta_i = +90^\circ$
- For a purely capacitive network: $\theta_v - \theta_i = -90^\circ$

- In a purely inductive and capacitive circuits, the average power is zero.
- Therefore, no transformation of energy from electric to nonelectric form takes place.
- The power is continually exchanged between the source driving the circuit and the electric or magnetic field associated with the elements.

Example 1

Calculate the average power at the terminals of the network

If $v = 100\cos(\omega t + 15^\circ)$ V, and $i = 4\sin(\omega t - 15^\circ)$ A.



Simple application of the formula we have just studied.

No!!! The current *i* is expressed in terms of the sine function!

The first step in the calculation for P is to rewrite *i* as a cosine, before using the formula.

$$i = 4\cos(\omega t - 105^\circ) \text{ A.}$$

$$P = \frac{1}{2}(100)(4)\cos[15 - (-105)] = -100 \text{ W}$$

The negative value of means that the network inside the box is delivering average power to the terminals.

Maximum Average Power Transfer 1

In Linear Circuit A we established that to **maximise power delivered to a load** in a resistive only circuit, the condition was that $R_L=R_{TH}$, where R_{TH} is the equivalent resistance of the circuit.

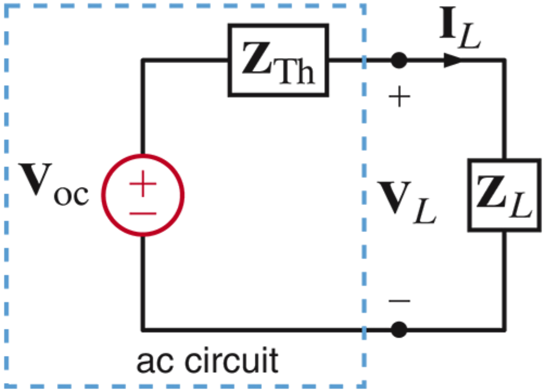
Let us now look at this more generic case with (complex) impedances.

- The average power transferred to the load is $P_L = \frac{1}{2} V_L I_L \cos\left(\theta_{v_L} - \theta_{i_L}\right)$

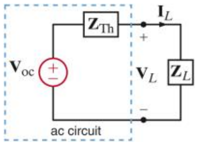
- For this circuit it holds that

$$\mathbf{I}_L = \frac{\mathbf{V}_{oc}}{Z_{Th} + Z_L} \qquad \mathbf{V}_L = \frac{\mathbf{V}_{oc} Z_L}{Z_{Th} + Z_L}$$

$$\mathbf{Z}_{Th} = R_{Th} + jX_{Th} \qquad \mathbf{Z}_L = R_L + jX_L$$



Maximum Average Power Transfer 2



Current and voltage at the load are complex numbers, we want to calculate their <u>absolute values</u>. Let us start from I_L.

$$I_L = \frac{V_{oc}}{Z_{th} + Z_L} = \frac{V_{oc}}{(R_{th}+R_L) + j(X_{th} + X_L)} \rightarrow |\boldsymbol{I_L}|?$$

Before calculating the absolute value, you need to **rationalise** the expression (i.e., no complex numbers at the denominator!). Please familiarize yourself with rationalization of complex functions!

$$I_L = \frac{V_{oc}}{(R_{th}+R_L) + j(X_{th} + X_L)} \frac{(R_{th}+R_L) - j(X_{th} + X_L)}{(R_{th}+R_L) - j(X_{th} + X_L)} = \frac{V_{oc}[(R_{th}+R_L) - j(X_{th} + X_L)]}{[(R_{th}+R_L)^2 + (X_{th}+X_L)^2]}$$

Look at this expression. The numerator is written as real & imaginary part, the denominator is real. At this stage you can calculate absolute value by definition (square root of the squared real part plus squared imaginary part).

$$|\boldsymbol{I_L}| = |V_{oc}| \sqrt{\left[\frac{(R_{th}+R_L)}{(R_{th}+R_L)^2 + (X_{th}+X_L)^2}\right]^{\mathbf{2}} + \left[\frac{-(X_{th} + X_L)}{(R_{th}+R_L)^2 + (X_{th}+X_L)^2}\right]^{\mathbf{2}}} = |V_{oc}| \sqrt{\frac{[(R_{th}+R_L)^2 + (X_{th}+X_L)^2]}{[(R_{th}+R_L)^2 + (X_{th}+X_L)^2]^{\mathbf{2}}}} =$$

$$= \frac{|\boldsymbol{V_{oc}}|}{\sqrt{[(\boldsymbol{R_{th}}+\boldsymbol{R_L})^{\mathbf{2}} + (\boldsymbol{X_{th}}+\boldsymbol{X_L})^{\mathbf{2}}]}}$$

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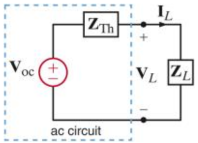
Maximum Average Power Transfer 3

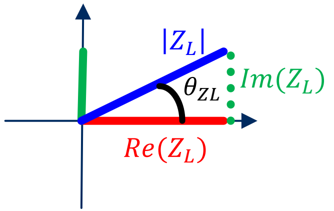


What about the magnitude of the voltage? We can use the generalized concept of Ohm's law with impedances.

$$|I_L| = \frac{|V_{oc}|}{\sqrt{[(R_{th}+R_L)^2 + (X_{th}+X_L)^2]}} \rightarrow |\boldsymbol{V_L}| = |Z_L I_L| = |Z_L||I_L| = \frac{|\boldsymbol{V_{oc}}|\sqrt{\boldsymbol{R_L^2} + \boldsymbol{X_L^2}}}{\sqrt{[(\boldsymbol{R_{th}}+\boldsymbol{R_L})^2 + (\boldsymbol{X_{th}}+\boldsymbol{X_L})^2]}}$$

What is missing now to calculate the average power on the load, is the factor $\cos(\theta_{VL} - \theta_{IL})$. We can demonstrate that this is the phase of the impedance of the load, in fact:

$$V_L = Z_L I_L \text{ in polar form} \rightarrow |V_L|e^{j\theta_{VL}} = |Z_L|e^{j\theta_{ZL}}|I_L|e^{j\theta_{IL}} \text{ only the phases} \rightarrow \theta_{VL} = \theta_{ZL} + \theta_{IL} \rightarrow \theta_{VL} - \theta_{IL} = \theta_{ZL}$$

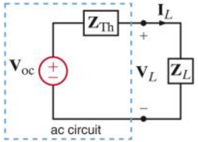


So we now want to find $\cos(\theta_{ZL})$. Let us think of the complex number Z_L in the complex space.

From trigonometry we can write that $\boldsymbol{\cos(\theta_{ZL})} = \frac{Re(Z_L)}{|Z_L|} = \frac{\boldsymbol{R_L}}{\sqrt{\boldsymbol{R_L^2}+\boldsymbol{X_L^2}}}$

Now we need to bring together all the pieces in the formula of the average power.

Maximum Average Power Transfer 4



Now we need to bring together all the pieces in the formula of the average power P.

$$P = \frac{1}{2}|V_L||I_L|\,cos(\theta_{ZL}) = \frac{1}{2}\frac{|V_{oc}|\sqrt{R_L^2 + X_L^2}}{\sqrt{[(R_{th}+R_L)^2 + (X_{th}+X_L)^2]}}\frac{|V_{oc}|}{\sqrt{[(R_{th}+R_L)^2 + (X_{th}+X_L)^2]}}\frac{R_L}{\sqrt{R_L^2 + X_L^2}} = \frac{1}{2}\frac{|V_{oc}|^2 R_L}{(R_{th}+R_L)^2 + (X_{th}+X_L)^2}$$

How to maximize this expression as a function of the load parameters R_L and X_L? Mathematically (see the book) you can calculate the partial derivatives with respect to R_L and X_L and set them to zero to derive two conditions.

Trusting the derivations of the book, we finally obtain the conditions that we looked for to maximize the average power delivered to the load, i.e. $X_{th} = -X_L$ and $R_L = \sqrt{R_{th}^2 + (X_{th}+X_L)^2}$

Combining the two equations above we get that $\boldsymbol{Z_L = Z_{th}^*}$ so the impedance of the load is equal to the complex conjugate of the Thevenin impedance! Key result!

In that case the maximum average power is $\boldsymbol{P_{max}} = \frac{1}{2}\frac{|V_{oc}|^2 R_L}{(R_{th}+R_L)^2} = \frac{1}{2}\frac{|V_{oc}|^2 R_{th}}{(2R_{th})^2} = \boldsymbol{\frac{|V_{oc}|^2}{8R_{th}}}$

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Maximum Average Power Transfer 5

What happens **if the load is purely resistive**? There is no load reactance $X_L = 0$, i.e. $Z_L = R_L$

The two conditions for maximizing the average power become:

$X_L = 0$ (purely resistive load) and $\boldsymbol{R_L} = \sqrt{R_{th}^2 + {X_{th}}^2} = |\boldsymbol{Z_{th}}|$

Essentially the resistance (or impedance, they are the same in this case because there is no load reactance) of the load is equal to the magnitude of the Thevenin impedance. Another key result!

Coffee Break

Effective or RMS value

Effective or RMS Value 1

- We have seen that the average absorbed power depends on the type(s) of sources that supply the power

  - DC (in Linear Circuit A) → $I^2 R$
  - Sinusoidal (with the assumption of purely resistive network) → $(1/2)I_M^2 R$

- But these are not the only possibilities, as in general the source can assume any periodic shape as a function of time -> We should like a general comparison instrument.

- Consequently, we define the **effective** value of <u>any periodic</u> source function, a tool to compare the power delivered by this function to the power delivered by a reference DC source.

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Effective or RMS Value 2

- We consider the periodic (not necessarily sinusoidal) current $i(t)$
- The average power delivered to a resistance *R* by this current is $P = \frac{1}{T}\int_0^T R\, i^2(t)dt$
- The effective value of the current, I_{eff}, is the DC current that delivers the same average power P to a resistor compared to the periodic current, so $P = RI_{eff}^2$
- Equating the two values of the power:

$$I_{eff}^2 R = \frac{1}{T}\int_{t_0}^{t_0+T} i^2(t) R dt \Longrightarrow I_{eff} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} i^2(t) dt}$$

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Effective or RMS Value 3

$$I_{eff} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} i^2(t)\,dt}$$

Important to note that the effective value is calculated via an **RMS** (Root Mean Square) operation, so effective or RMS values are synonyms.

- We firstly take the square of the instantaneous current: square
- Then we average: mean
- And finally we take the square root: root

- A crucial result is that for sinusoidal sources the RMS/effective value is equal to the maximum (peak) value divided by the square root of 2.

$$I_{eff} = I_{rms} = \frac{I_M}{\sqrt{2}}$$

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Example 2

A sinusoidal voltage with a maximum amplitude of 625 V is applied to the terminals of a 50 Ohm resistor. What is the average power delivered to the resistor?

Use RMS value as it is a sinusoidal source.

The RMS value of the sinusoidal voltage is approximately $\frac{625}{\sqrt{2}}$ =441.94 V

Hence $P = \frac{V_{eff}^2}{R}$

$$P = \frac{(441.94)^2}{50} = 3906.25 \text{ W}.$$

Use the average power formula for purely resistive networks (slide 12). $P = \frac{1}{2} V_M I_M$

$$P = \frac{1}{2} V_M I_M = \frac{1}{2} V_M \left(\frac{V_M}{R}\right) = \frac{1}{2} * 625 * \frac{625}{50} = 3906.25\, W$$

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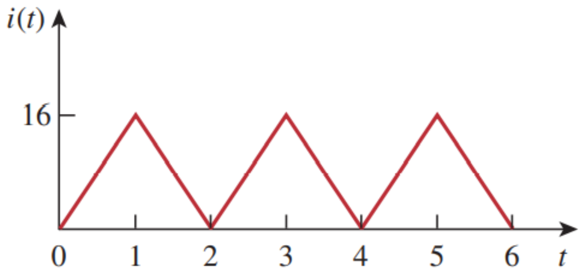
Example 3a

Practice Problem 11.7:

- Find the RMS value of the current waveform.



$$i(t) = \begin{cases} 16t & 0 < t < 1 \\ 32 - 16t & 1 < t < 2 \end{cases}$$

$$T = 2$$

$$I_{rms}^2 = \frac{1}{T}\int_0^T i^2\,dt = \frac{1}{2}\left[\int_0^1 (16t)^2\,dt + \int_1^2 (32-16t)^2\,dt\right]$$

$$I_{rms}^2 = \frac{256}{2}\left[\int_0^1 t^2\,dt + \int_1^2 (4-4t+t^2)\,dt\right]$$

$$I_{rms}^2 = 128\left[\frac{1}{3} + \left(4t - 2t^2 + \frac{t^3}{3}\right)\Bigg|_1^2\right] = \frac{256}{3}$$

$$I_{rms} = \sqrt{\frac{256}{3}} = \mathbf{9.238\ A}$$

Capabilities:
-Write expression of waveforms
-Integration

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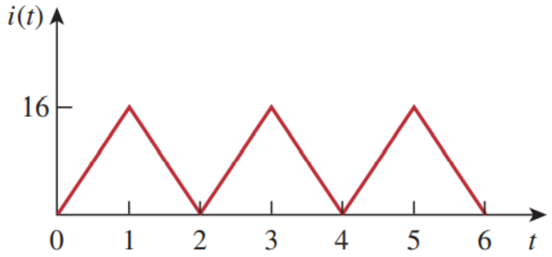
Example 3b



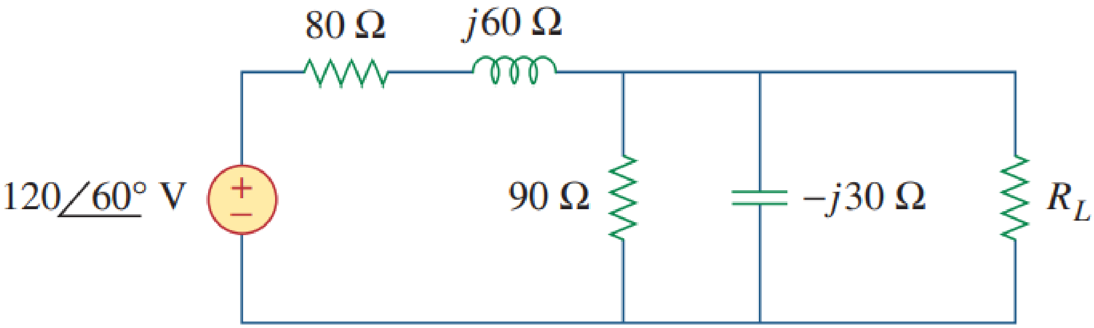
Practice Problem 11.7:

- If the current flows through a 9 Ohm resistor, calculate the average power absorbed by the resistor.

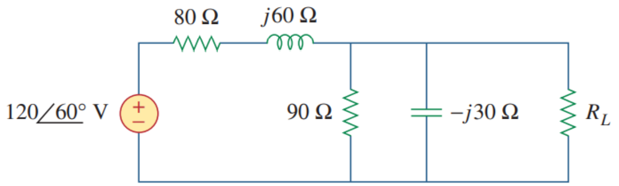
$$P = I_{rms}^2 R = (9.238^2)(9) = \mathbf{768\ w}$$

Example 4a

Practice Problem 11.6: The load resistor is adjusted until it absorbs the maximum average power. Calculate its value R_L and the maximum average power absorbed by it.



Example 4b



Maximizing the power implies adjusting the load to the Thevenin impedance of the circuit, which needs to be computed:

$$\mathbf{Z}_1 = 80 + j60$$

$$\mathbf{Z}_2 = 90 \,||\, (-j30) = \frac{(90)(-j30)}{90 - j30} = 9(1 - j3)$$

$$\mathbf{Z}_{Th} = \mathbf{Z}_1 \,||\, \mathbf{Z}_2 = \frac{(80 + j60)(9 - j27)}{80 + j60 + 9 - j27} = 17.181 - j24.57\ \Omega$$

$$R_L = \left| \mathbf{Z}_{Th} \right| = \mathbf{30\ \Omega}$$

Recall the formula of avg power maximization for purely resistive loads!

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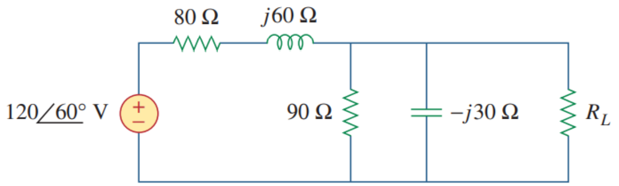
Example 4c



Calculating the Thevenin voltage of the circuit seen by the load:

$$\mathbf{V}_{\text{Th}} = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}(120\angle 60°) = \frac{(9)(1 - j3)}{89 + j33}(120\angle 60°)$$

$$\mathbf{V}_{\text{Th}} = 35.98\angle -31.91°$$

The current through the load:

$$\mathbf{I} = \frac{\mathbf{V}_{\text{Th}}}{\mathbf{Z}_{\text{Th}} + \text{R}_\text{L}} = \frac{35.98\angle -31.91°}{47.181 - j24.57} = 0.6764\angle -4.4°$$

The maximum average power absorbed by the load:

$$\text{P}_{\max} = \frac{1}{2}|\mathbf{I}|^2 \text{R}_\text{L} = \frac{1}{2}(0.6764)^2(30) = \mathbf{6.863\ W}$$

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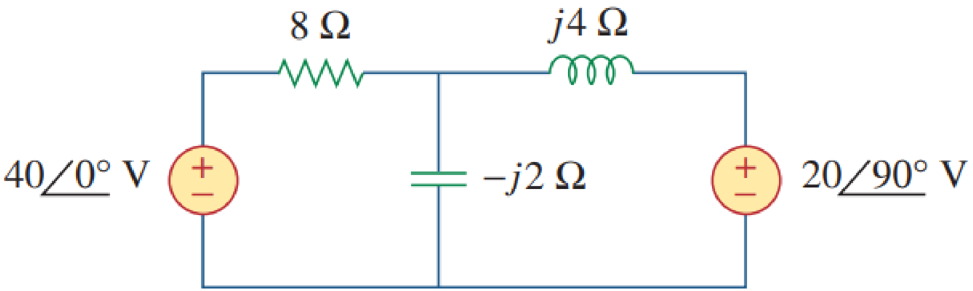
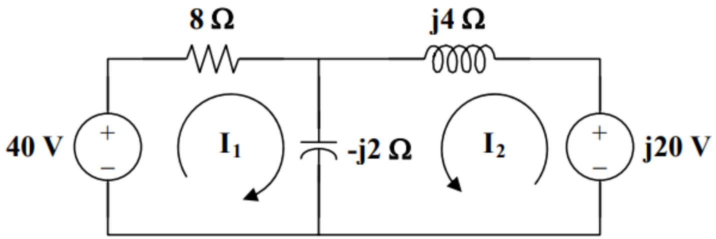
Example 5a

Practice Problem 11.14: Calculate the average power absorbed by each of the five elements in the circuit.



What is the average absorbed power of the capacitor and the inductor first of all?

Example 5b



You need voltages and currents for each element.

For mesh 1: $-40 + (8 - j2)\mathbf{I}_1 + (-j2)\mathbf{I}_2 = 0$

$(4 - j)\mathbf{I}_1 - j\mathbf{I}_2 = 20$

$\mathbf{I}_1 = 5\angle 53.14°$

$\mathbf{I}_2 = 13.6\angle 17.11°$

For mesh 2: $-j20 + (j4 - j2)\mathbf{I}_2 + (-j2)\mathbf{I}_1 = 0$

$-j\mathbf{I}_1 + j\mathbf{I}_2 = j10$

For the 40-V voltage source:

$$\mathbf{V}_s = 40\angle 0°$$

$$\mathbf{I}_1 = 5\angle 53.14°$$

$$P_s = \frac{-1}{2}(40)(5)\cos(-53.14°) = \mathbf{-60\ W}$$

For the j20-V voltage source:

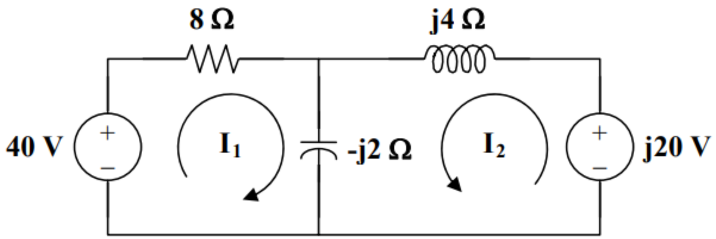
$$\mathbf{V}_s = 20\angle 90°$$

$$\mathbf{I}_2 = 13.6\angle 17.11°$$

$$P_s = \frac{-1}{2}(20)(13.6)\cos(90° - 17.11°) = \mathbf{-40\ W}$$

Watch out for the **minus signs** in the power expression coming from the passive sign convention. The defined mesh currents are entering the sources into their negative polarities!

Example 5c



For the resistor:

$$I = |\mathbf{I}_1| = 5$$

$$V = 8|\mathbf{I}_1| = 40$$

$$P = \frac{1}{2}(40)(5) = \mathbf{100\ W}$$

The average power absorbed by the inductor and capacitor is zero watts. (Do you remember why?)

Apparent Power (not for the exam)

Not part of the exam programme, but you have to know that these quantities (complex power *S* and apparent power) exist – if in later courses about energy conversion you need to look at these concepts, please refer to chapter 11 of the book.

Complex Power

$$S = P + jQ = V_{rms}(I_{rms})^* = |V_{rms}||I_{rms}|\angle(\theta_v - \theta_i)$$

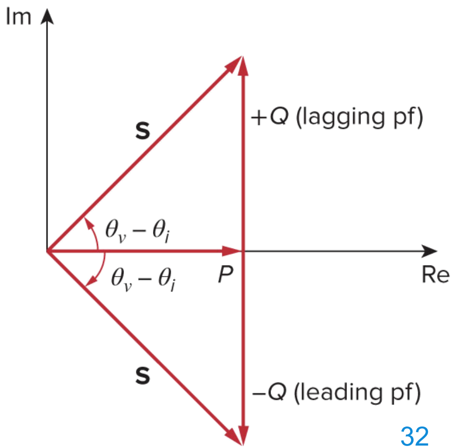
Apparent Power

$$S = |S| = |V_{rms}||I_{rms}| = \sqrt{P^2 + Q^2} \text{ [VA]}$$

Real Power $P = \mathrm{Re}(S) = S\cos(\theta_v - \theta_i)$ [Watt]

Reactive Power $Q = \mathrm{Im}(S) = S\sin(\theta_v - \theta_i)$ $[VAR]$

Power Factor= $\frac{P}{S} = \cos(\theta_v - \theta_i)$



Summary of the day

- Instantaneous power, average power
- Maximum average power transfer
- Effective or RMS value

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Next steps

- **SGH** (Self-Graded Homework) assignments.
- **Seminar**: Tuesday & Friday.
- **Next week**: **Mid-term, best of luck!**
- Week 2.6 -> Magnetically coupled circuits, ideal transformer

Thank you!

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