EE1C2 “Linear Circuits B”

Week 2.6

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TUDelft

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Faculty of Electrical Engineering,
Mathematics and Computer Science

Today

- Recapitulation of weeks 2.1–2.4
- Magnetically coupled circuits
  - introduction of the concept, circuit analysis
  - coupling coefficient
  - linear / ideal transformers
- Summary and conclusions
- Next tasks

Recap of weeks 2.1–2.4

- Steady-state sinusoidal systems
  - phasor-domain quantities, impedance, admittance, circuit analysis
- Transfer functions
  - standard form, poles, zeros
  - Bode plots
- Filters and resonance
- Power in AC
  - instantaneous & average powers, RMS values
  - maximum power transfer → conjugate matching

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Self & mutual inductance

Self & mutual inductance

- We now enter the electromagnetics territory → the relevant phenomena will be explained in EE1P1 "Electricity & Magnetism" and EE2P1 "Electromagnetics"
- We can examine configurations with Linear Circuits instruments as long as:
  - all intervening ingredients are linear
  - the physical dimension of the electrical elements are small with respect to the wavelengths in the electromagnetic field
- As always in this course, we consistently apply the passive sign convention (this will be different in EE1P1)

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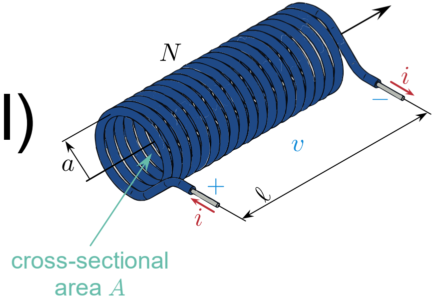
Self & mutual inductance

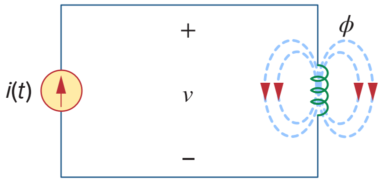
- Basic principle:
  - a time-varying current $i(t)$ → a time-varying magnetic flux $\phi(t)$
  - a time-varying magnetic flux $\phi(t)$ → an induced voltage $v(t)$

- The most common situation: an inductor (coil)



- A voltage can be induced:
  - in the fed inductor itself → self-inductance
  - in another nearby located inductor → mutual inductance

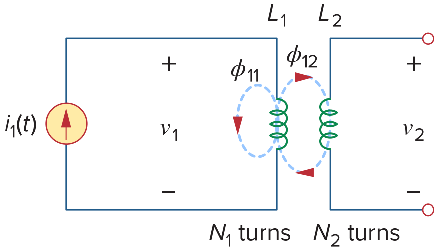
Self & mutual inductance



- Consider a single inductor with N turns
- The induced voltage is in this case: $v(t) = N\frac{\mathrm{d}\phi}{\mathrm{d}t}$
- Assuming all media are linear $\phi(t) = L\,i(t)$ with L being a constant denoted as self-inductance
- The voltage induced by the time-varying current in the same inductor is then:

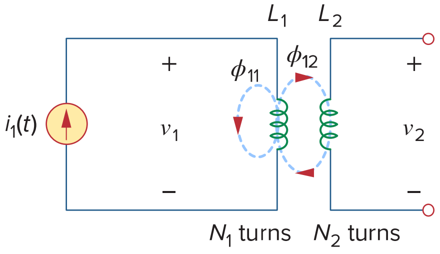
$$v(t) = L\frac{\mathrm{d}i}{\mathrm{d}t}$$

Self & mutual inductance



- Consider now two nearby-located inductors with N_1 and N_2 turns, respectively
- Only inductor 1 is fed, with inductor 2 being open
- The inductors' self-inductances are L_1 and L_2, respectively
- The magnetic flux produced by inductor 1 is $\phi_1 = \phi_{11} + \phi_{12}$:
  - ϕ_{11} links to the inductor 1
  - ϕ_{12} links to inductor 2
- We now analyse the induced voltages $v_1(t)$ and $v_2(t)$

Self & mutual inductance



- According to the definition of the self-inductance: $v_1(t) = L_1 \dfrac{\mathrm{d}i_1}{\mathrm{d}t}$
- The flux ϕ_{12} links to the N_2 turns of inductor 2
- By analogy: $v_2(t) = N_2 \dfrac{\mathrm{d}\phi_{12}}{\mathrm{d}t} = M_{21} \dfrac{\mathrm{d}i_1}{\mathrm{d}t}$

  with M_{21} being a constant denoted as mutual inductance
- Similarly, if inductor 2 is fed:

$$v_2(t) = L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t} \qquad v_1(t) = N_1 \frac{\mathrm{d}\phi_{21}}{\mathrm{d}t} = M_{12} \frac{\mathrm{d}i_2}{\mathrm{d}t}$$

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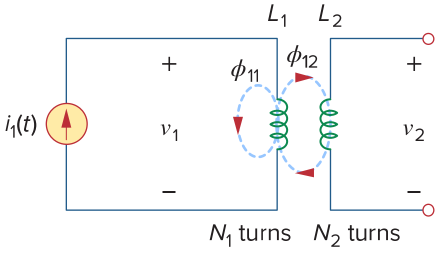
Self & mutual inductance



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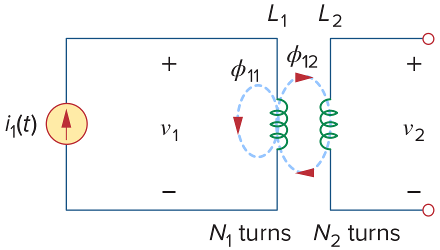
source

receiver

Self & mutual inductance



Important observations:

- The self-inductance is always positive → dictated by the passive sign convention
- The mutual coupling depends on the orientation of the inductors:
  - the same pair of inductors 1 and 2
  - at specified (fixed) locations
  - the same producing current i_1
  - the receiver 2 parallel or anti-parallel

} the induced voltage can be v_2 or $-v_2$!

↓

the mutual coupling can be M_{21} or $-M_{21}$!

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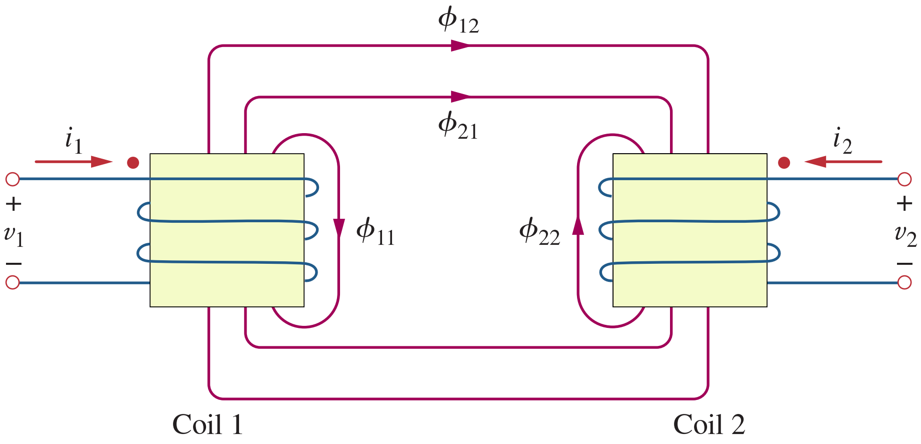
The dot convention

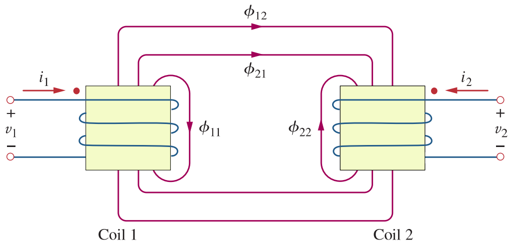
The dot convention

- Accounting for the relative position between the inductors is achieved via the dot convention
- The dot indicates the "start" of the windings and is associated with a given (conventional) way of winding the wire



- One must always correlate the current's direction and the "dot"

The dot convention



Rules for using the dots:

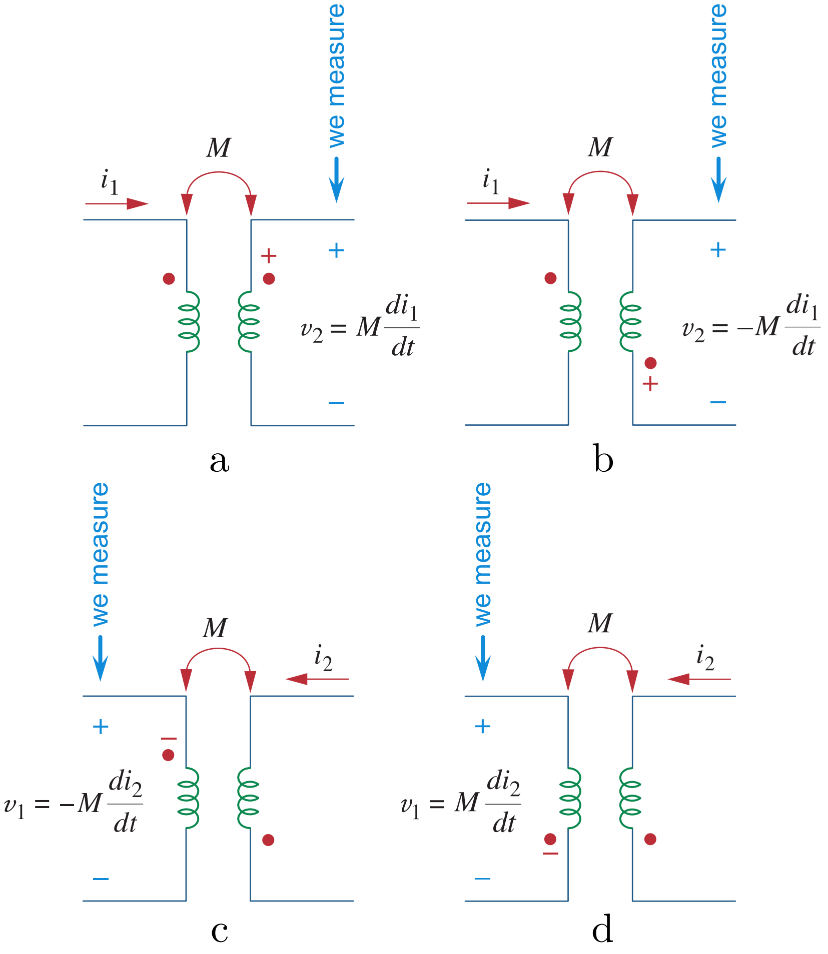
- If a current enters the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is positive at the dotted terminal of the second coil.
- If a current leaves the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is negative at the dotted terminal of the second coil.

The dot convention

Typical cases:

- If a current enters the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is positive at the dotted terminal of the second coil.
- a & b: i_1 enters the dotted terminal ⟶ v_2 has the **+** at the dot

- If a current leaves the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is negative at the dotted terminal of the second coil.
- c & d: i_2 leaves the dotted terminal ⟶ v_1 has the **-** at the dot



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The dot convention: inductors in series

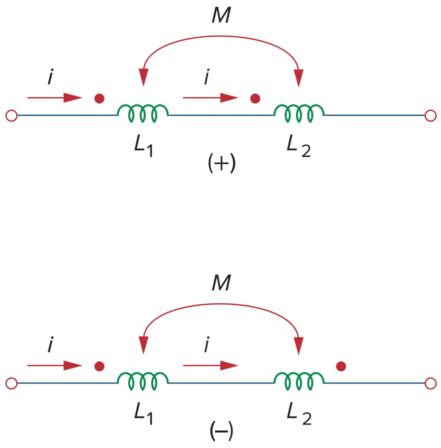
Mutually coupled inductors connected in series:

- Series-adding connection: the current enters both dotted terminals

$$L = L_1 + L_2 + 2M$$

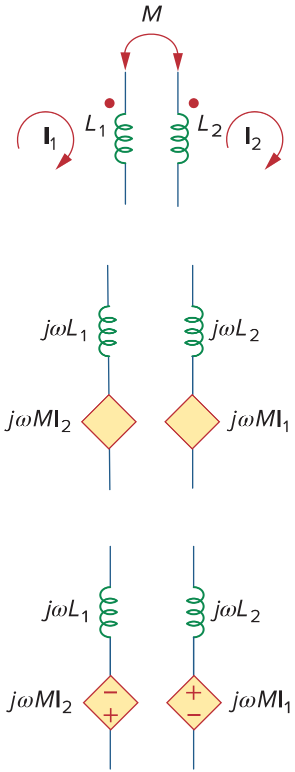
- Series-opposing connection: the current enters one dotted terminal and leaves the other dotted terminal

$$L = L_1 + L_2 - 2M$$



The dot convention: practical strategy

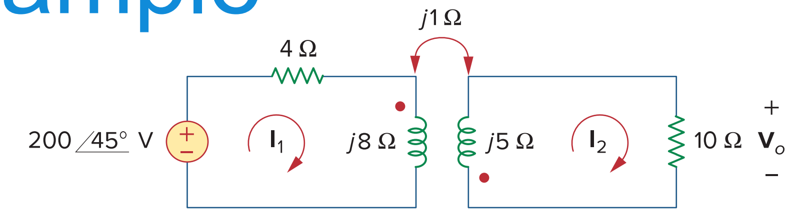
1) Start from the given circuit
2) Fill in the impedance corresponding to the self-inductance
3) Insert a voltage-dependent voltage source for each coupling → fill in the value of the corresponding induced voltage (no sign!)
4) Assign the polarity of the sources based on the dot convention
5) Solve the new phasor-domain circuit – it is now devoid of any magnetic couplings!



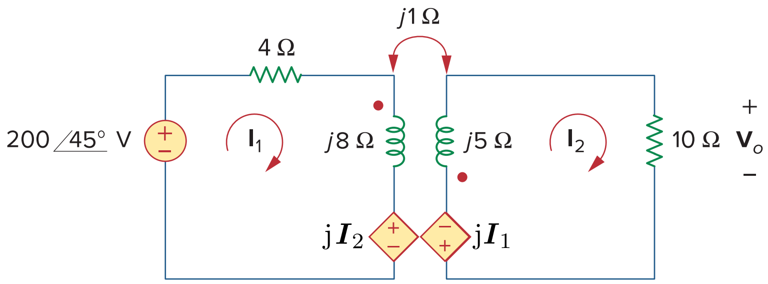
The dot convention: example

- Determine the voltage V_o in the circuit at the right

  (Practice Problem 13.1)



- Redraw the circuit with dependent sources:
  1) place the dependent sources and fill in the relevant values
  2) decide the polarity



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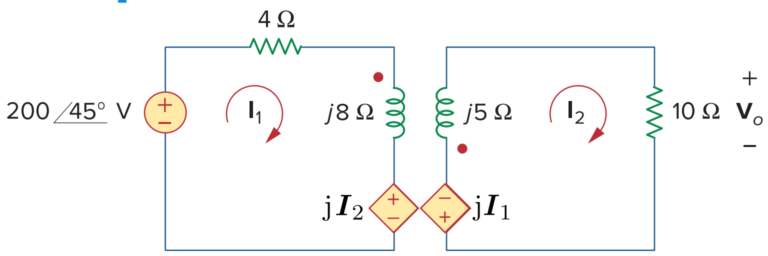
The dot convention: example



- Mesh equations:

$$(4 + \mathrm{j}8)\boldsymbol{I}_1 + \mathrm{j}\boldsymbol{I}_2 = 200\angle 45^\circ$$

$$\mathrm{j}\boldsymbol{I}_1 + (10 + \mathrm{j}5)\boldsymbol{I}_2 = 0$$

- Solution of the system:

$$\boldsymbol{I}_2 = \frac{200\angle 45^\circ}{-100 + j}$$

- Final solution:

$$\boldsymbol{V}_\mathrm{o} = 10\boldsymbol{I}_2 = \frac{10 \cdot 200\angle 45^\circ}{-100 + j} = 20\angle - 135^\circ$$

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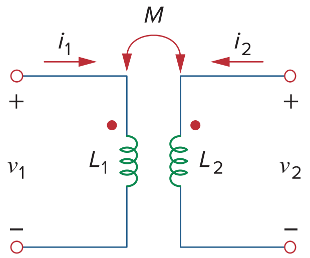
Energy in coupled circuits

Energy in coupled circuits



- Energy stored in coupled inductor m:

$$w_m = \int_{\tau=t_1}^{t_2} v_m(\tau) i_m(\tau) \mathrm{d}\tau$$

  – with the voltage v_m being: $v_m(\tau) = v_{mm}(\tau) + v_{mn}(\tau)$

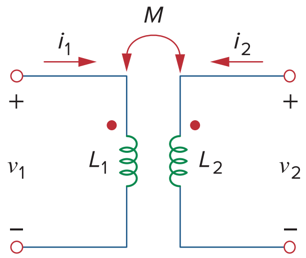
  $v_{mm}(\tau) = L_m \frac{\mathrm{d}i_m}{\mathrm{d}\tau}$, $v_{mn}(\tau) = M_{mn} \frac{\mathrm{d}i_n}{\mathrm{d}\tau}$

- Initial state: $i_1 = 0$ and $i_2 = 0$
- Step 1: vary i_1 from 0 to I_1 and keep $i_2 = 0$

  voltages in the circuit: $v_1(\tau) = L_1 \frac{\mathrm{d}i_1}{\mathrm{d}\tau}$ and $v_2(\tau) = 0$
- Step 2: vary i_2 from 0 to I_2 and keep $i_1 = I_1$

  voltages in the circuit: $v_1(\tau) = M_{12} \frac{\mathrm{d}i_2}{\mathrm{d}\tau}$ and $v_2(\tau) = L_2 \frac{\mathrm{d}i_2}{\mathrm{d}\tau}$

Energy in coupled circuits



- Step 1: energy just in inductor 1

$$w' = w_1 = \int_{\tau=0}^{t_1} v_1(\tau) i_1(\tau) \mathrm{d}\tau = \int_{\tau=0}^{t_1} L_1 \frac{\mathrm{d}i_1}{\mathrm{d}\tau} i_1(\tau) \mathrm{d}\tau = \frac{1}{2} L_1 I_1^2$$

- Step 2: energy in both inductors

$$w'' = w_1 = \int_{\tau=t_1}^{t_2} v_1(\tau) i_1(\tau) \mathrm{d}\tau = \int_{\tau=t_1}^{t_2} M_{12} \frac{\mathrm{d}i_2}{\mathrm{d}\tau} I_1 \mathrm{d}\tau = M_{12} I_2 I_1$$

$$w''' = w_2 = \int_{\tau=t_1}^{t_2} v_2(\tau) i_2(\tau) \mathrm{d}\tau = \int_{\tau=t_1}^{t_2} L_2 \frac{\mathrm{d}i_2}{\mathrm{d}\tau} i_2(\tau) \mathrm{d}\tau = \frac{1}{2} L_2 I_2^2$$

- Total energy:

$$w_{\text{tot}} = w' + w'' + w''' = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M_{12} I_2 I_1$$

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Energy in coupled circuits

- Repeating the steps in the reversed order (first 2, then 1) total energy:

$$w_{\text{tot}} = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M_{21} I_1 I_2$$

- Since the two energies are the same $\longrightarrow$ $M_{12} = M_{21} = M$

- In general, the coupling can be positive or negative:

$$w = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M I_1 I_2$$

positive coupling

$$w = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 - M I_1 I_2$$

negative coupling

Energy in coupled circuits

- The system is passive → energy must be positive

(passive sign convention!)

$$\frac{1}{2}L_1I_1^2 + \frac{1}{2}L_2I_2^2 - MI_1I_2 \geqslant 0$$

- A bit of algebra → $M \leqslant \sqrt{L_1L_2}$

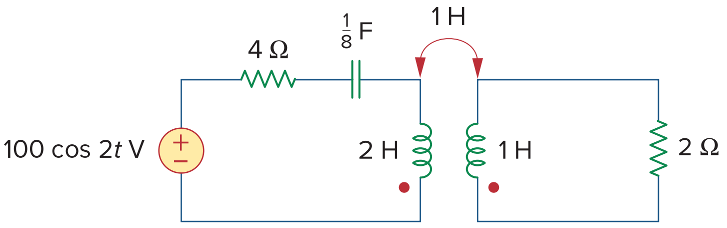
- We define the coupling coefficient with $0 \leqslant k \leqslant 1$

$$k = \frac{M}{\sqrt{L_1L_2}}$$

Energy in coupled circuits: example

- Determine the coupling coefficient in the circuit at the right, and the energy stored in the coupled inductors at $t = 1.5\,\mathrm{s}$.

  (Practice Problem 13.3)

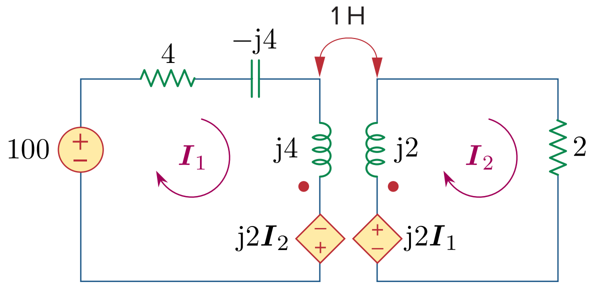


- Coupling coefficient:

$$k = \frac{M}{\sqrt{L_1 L_2}} = \frac{1}{\sqrt{2 \cdot 1}} = 0.7071$$

Energy in coupled circuits: example

1) Transform the circuit to the phasor domain
2) Replace the coupling by dependent sources and fill in the relevant values
3) Decide the polarity
4) The rest is known



Mesh analysis

$$\begin{cases} 100 = (4 - j4 + j4)\boldsymbol{I}_1 - j2\boldsymbol{I}_2 \\ -j2\boldsymbol{I}_1 + (2 + j2)\boldsymbol{I}_2 = 0 \end{cases} \Rightarrow \begin{cases} \boldsymbol{I}_2 = 13.87\angle 56.31^\circ \Rightarrow i_2 = 13.87\cos(2t + 56.31^\circ)\,(\mathrm{A}) \\ \boldsymbol{I}_1 = 19.66\angle 11.31^\circ \Rightarrow i_1 = 19.66\cos(2t + 11.31^\circ)\,(\mathrm{A}) \end{cases}$$

$$w = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 - M(i_1 i_2)$$

$$= \frac{1}{2}2\,(-19.62)^2 + \frac{1}{2}1\,(-9.25)^2 - 1\,(-19.62)(-9.25) = 246.2\,(\mathrm{J})$$

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Linear transformers

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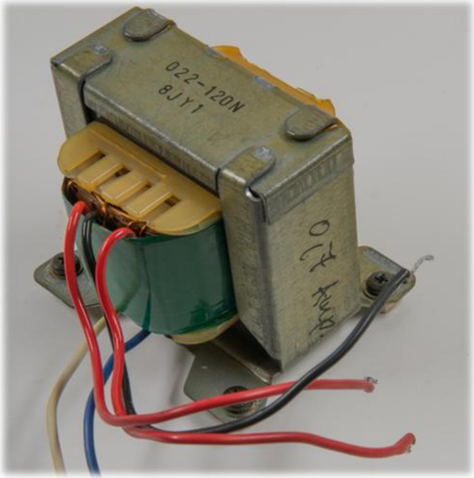
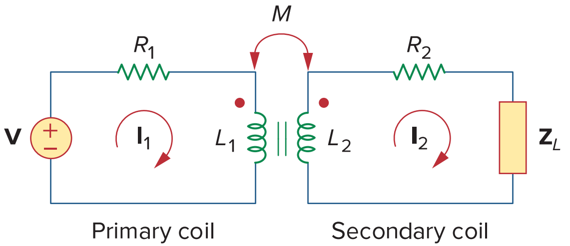
Linear transformer

- A four-terminal device comprising two (or more) magnetically coupled coils
- A direct application to mutual induction



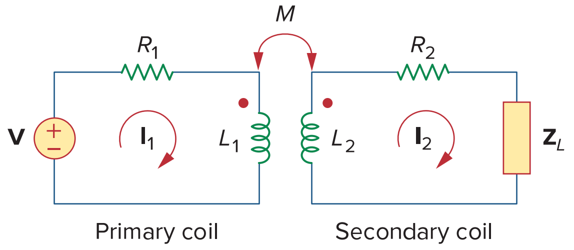
Linear transformers



- Main elements:
  - primary coil: the inductor connected to the source
  - secondary coil: the inductor connected to the load
  - (magnetic) core: a medium enhancing the mutual coupling

- We only consider linear transformers → the relation between the voltages/currents pertaining to the primary and secondary sections is linear

Linear transformers



- Determine the impedance in the primary section

$$\boldsymbol{V} = (R_1 + j\omega L_1)\boldsymbol{I}_1 - j\omega M \boldsymbol{I}_2$$

$$0 = -j\omega M \boldsymbol{I}_1 + (R_2 + j\omega L_2 + \boldsymbol{Z}_L)\boldsymbol{I}_2$$

- The input impedance: $\boldsymbol{Z}_{in} = \dfrac{\boldsymbol{V}}{\boldsymbol{I}_1} = R_1 + j\omega L_1 + \dfrac{\omega^2 M^2}{R_2 + j\omega L_2 + \boldsymbol{Z}_L}$
- Elements:
  - primary impedance: $R_1 + j\omega L_1$
  - reflected impedance: $\boldsymbol{Z}_R = \dfrac{\omega^2 M^2}{R_2 + j\omega L_2 + \boldsymbol{Z}_L}$

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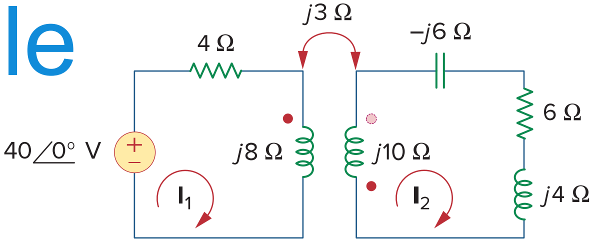
Linear transformers: example



$$\boldsymbol{V} = (R_1 + \mathrm{j}\omega L_1)\boldsymbol{I}_1 + \mathrm{j}\omega M \boldsymbol{I}_2$$
$$0 = +\mathrm{j}\omega M \boldsymbol{I}_1 + (R_2 + \mathrm{j}\omega L_2 + \boldsymbol{Z}_\mathrm{L})\boldsymbol{I}_2$$

- Find the input impedance in the circuit at the right, and the current drawn from the voltage source. (assume $\omega = 1$)
  (Practice Problem 13.4)
- Determine the input impedance:

$$\boldsymbol{Z}_\mathrm{in} = 4 + \mathrm{j}8 + \frac{3^2}{j10 - \mathrm{j}6 + 6 + \mathrm{j}4} = 4 + \mathrm{j}8 + \frac{9}{6 + j8}$$
$$= 8.58\angle 58.05^\circ$$

- Determine the current drawn from the voltage source

$$\boldsymbol{I} = \frac{\boldsymbol{V}}{\boldsymbol{Z}_\mathrm{in}} = \frac{40}{8.58\angle 58.05^\circ} = 4.66\angle -58.05^\circ$$

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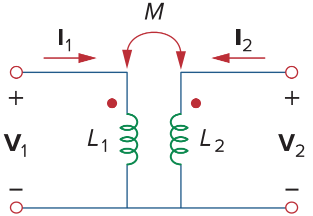
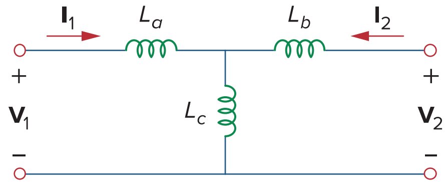
Linear transformers: equivalent circuits

- Purpose: replace the circuit with couplings with an equivalent circuit with no couplings



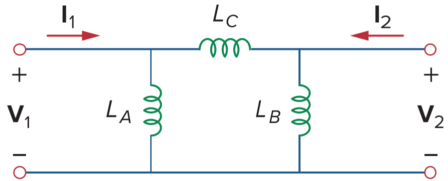
- Condition: the input/output voltages and currents do not change

T network



- Equivalent networks:
  - T network
  - Π network

Π network



Linear transformers: equivalent circuits

- T network

$$\begin{bmatrix} \boldsymbol{V}_1 \\ \boldsymbol{V}_2 \end{bmatrix} = \begin{bmatrix} j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_1 \\ \boldsymbol{I}_2 \end{bmatrix}$$

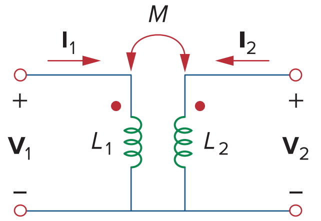
$$\begin{bmatrix} \boldsymbol{V}_1 \\ \boldsymbol{V}_2 \end{bmatrix} = \begin{bmatrix} j\omega(L_\mathrm{a} + L_\mathrm{c}) & j\omega L_\mathrm{c} \\ j\omega L_\mathrm{c} & j\omega(L_\mathrm{b} + L_\mathrm{c}) \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_1 \\ \boldsymbol{I}_2 \end{bmatrix}$$

  - equivalent inductance values

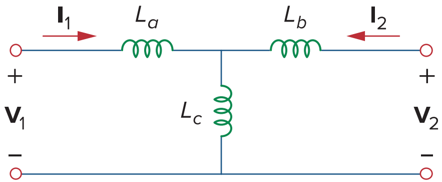
$$L_\mathrm{a} = L_1 - M \qquad L_\mathrm{b} = L_2 - M \qquad L_\mathrm{c} = M$$

- Π network

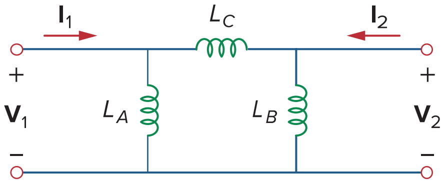
$$L_\mathrm{A} = \frac{L_1 L_2 - M^2}{L_2 - M} \qquad L_\mathrm{B} = \frac{L_1 L_2 - M^2}{L_1 - M} \qquad L_\mathrm{C} = \frac{L_1 L_2 - M^2}{M}$$



T network



Π network



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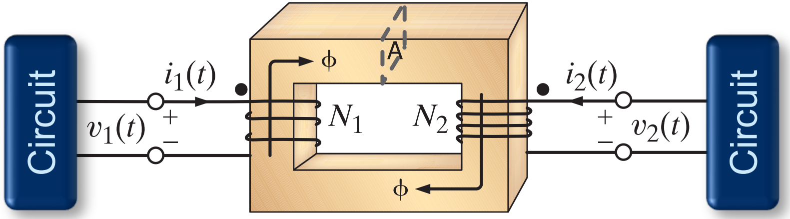
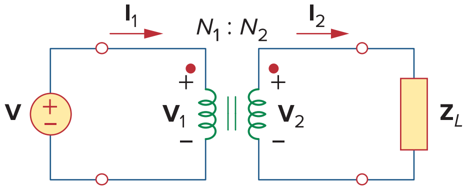
Ideal transformers

Ideal transformers



- Properties:
  - coupling coefficient $k = 1$
  - primary and secondary coils are lossless
  - primary and secondary coils have infinite self-inductances
- Transformers with an iron core are a good approximation of ideal transformers
- Standard notation:
  - N_1 : number of turns of the primary section
  - N_2 : number of turns of the secondary section
  - $n = N_2/\mathrm{N}_1$: turns ratio or transformation ratio

Ideal transformers



- Basic relations:
  - voltage ratio: $\dfrac{V_2}{V_1} = \dfrac{N_2}{N_1} = n$
  - current ratio: $\dfrac{I_2}{I_1} = \dfrac{N_1}{N_2} = \dfrac{1}{n}$
  - attention: magnitudes / rms values!

- Transformer types:
  - step-down transformer ($n < 1$): the secondary voltage is less than the primary one
  - step-up transformer ($n > 1$): the opposite

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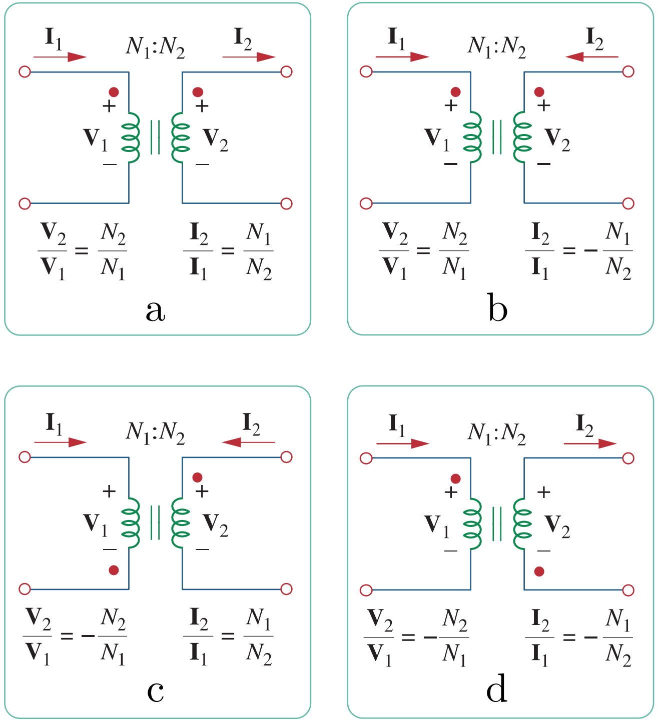
Ideal transformers

Basic relations accounting for the sign

- Voltage:
  - V_1 and V_2 are both positive or both negative at the dotted terminal, use $+n$ (cases a and b)
  - otherwise use $-n$ (cases c and d)
- Current:
  - I_1 and I_2 both enter or leave the dotted terminal, use $-1/n$ (cases b and d)
  - otherwise use $+1/n$ (cases a and c)



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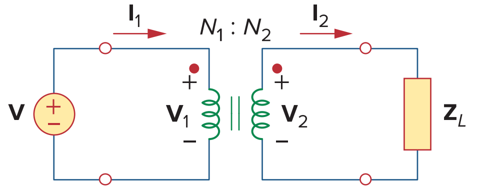
Ideal transformers



- Input impedance:

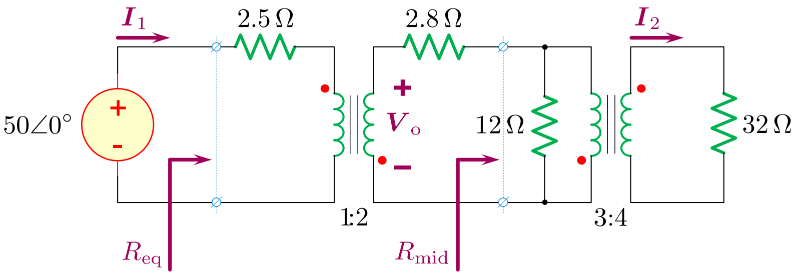
$$\boldsymbol{Z}_{\text{in}} = \frac{\boldsymbol{V}_1}{\boldsymbol{I}_1} = \frac{\boldsymbol{V}_2 \frac{N_1}{N_2}}{\boldsymbol{I}_2 \frac{N_2}{N_1}} = \frac{\boldsymbol{V}_2}{\boldsymbol{I}_2} \frac{1}{\left(\frac{N_2}{N_1}\right)^2} = \frac{\boldsymbol{Z}_{\text{L}}}{n^2}$$

Also denoted as reflected impedance

- Utility: one can remove an ideal transformer from the circuit, and capture the effect of the load and secondary stage back to the primary

Exam exercise example

Exam(ple)



- Consider the circuit at the right.

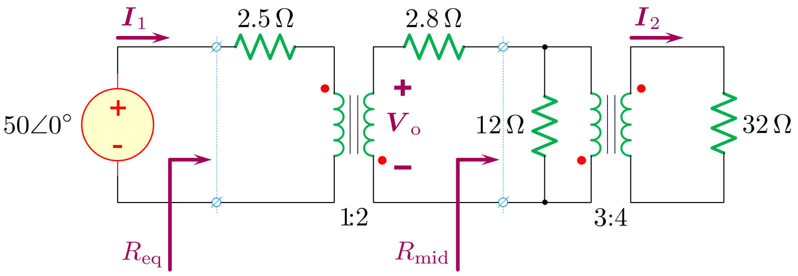
a) Calculate the equivalent resistance R_{mid}. (1 point)

b) Calculate the equivalent resistance R_{eq}. (1 point)

- Approach: Find the equivalent resistances by repeatedly applying the reflected impedances, working backwards from right to left

Exam(ple)

a) Calculate the equivalent resistance R_{mid}. (1 point)



- Reflect the $32\,\Omega$ resistance via the 3:4 transformer:

$$Z_{\text{ref}} = \frac{Z_{\text{L}}}{n^2} = \frac{32}{(4/3)^2} = 18\,\Omega$$

- Account for the parallel resistance:

$$12 \parallel 18 = \frac{12 \cdot 18}{12 + 18} = 7.2\,\Omega$$

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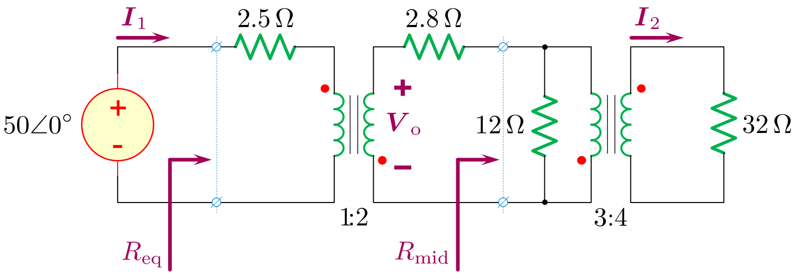
Exam(ple)



b) Calculate the equivalent resistance R_{eq}.

- Account for the $2.8\,\Omega$ series resistance:

  $$7.2 + 2.8 = 10\,\Omega$$

- Reflect the $10\,\Omega$ resistance via the 1:2 transformer:

  $$Z_{ref} = \frac{Z_L}{n^2} = \frac{10}{(2/1)^2} = 2.5\,\Omega$$

- Account for the series resistance:

  $$R_{eq} = 2.5 + 2.5 = 5\,\Omega$$

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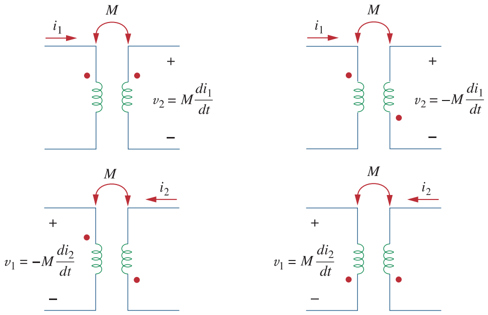
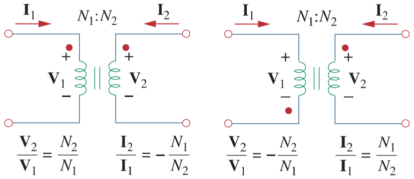
Summary of the day



- Magnetically coupled circuits
  - coupling coefficient
  - accounting via dot convention + dependent sources
- Linear transformer
- Ideal transformer
  - large inductances, perfect coupling, zero resistance
  - voltage/current transformation ratios
  - reflected impedance



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Next tasks

- Please do the SGH4
- Seminars of Tuesday and Friday
- Next week: a new transform (more extremely useful math!)

Thank you!

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